Ma341-001: Test #1
Answer Key

Friday, June 3, 2005
Instructor: Dr. Bill Cook

#1 (15 points) Classify the following equations:

(a) What kind of differential equation is "$y' = x - yx$"?
separable, linear, neither, or both

Notice that we can factor out an "x" so that $y' = x(1-y)$. Thus this equation is separable. Also, the equation can be rewritten as follows: $y' + xy = x$. Thus this equation is linear (in fact it is a first order nonhomogeneous linear equation).

Answer (a): The differential equation $y' = x - yx$ is *both* separable and linear.

(b) What kind of differential equation is "$y\,y' = \ln(x+y)$"?
separable, linear, neither, or both

The right hand side of the equation is given by the single term "$\ln(x+y)$". There is no "law of logarithms" to split this up. This tells us that the equation is not separable. This term also tells us that the equation is not linear.

Answer (b): The differential equation $y\,y' = \ln(x+y)$ is *neither* separable nor linear.

(c) What kind of differential equation is "$(e^{x+y} - \sin(x))\,\mathrm{d}x + (e^{x+y} + 2y)\,\mathrm{d}y = 0$"?
linear, exact, neither, or both

If we divide the equation by $\mathrm{d}x$ we get, $e^{x+y} - \sin(x) + (e^{x+y} + 2y)\,y' = 0$. Notice that e^{x+y} does not factor out, so we cannot get rid of it. This tells us that the equation is not linear.
To check exactness we look at the "mixed partials"

$$\frac{\partial}{\partial y}\left(e^{x+y} - \sin(x)\right) = e^{x+y} + 0$$

and

$$\frac{\partial}{\partial x}\left(e^{x+y} + 2y\right) = e^{x+y} + 0.$$

These are the same. Thus the equation is exact.

Answer (c): The differential equation $(e^{x+y} - \sin(x))\,\mathrm{d}x + (e^{x+y} + 2y)\,\mathrm{d}y = 0$ is *exact* but not linear.

#2 (18 points) A 10 liter tank is initially filled with pure water. A brine with a concentration of 2 kilograms of salt per liter is pumped into the tank at a rate of 5 liters per minute. The brine in the tank is well mixed and drained at a rate of 5 liters per minute. Let $x(t)$ denote the amount of salt (in kilograms) in the tank after t minutes have gone by.

(a) State an initial value problem which $x(t)$ satisfies.

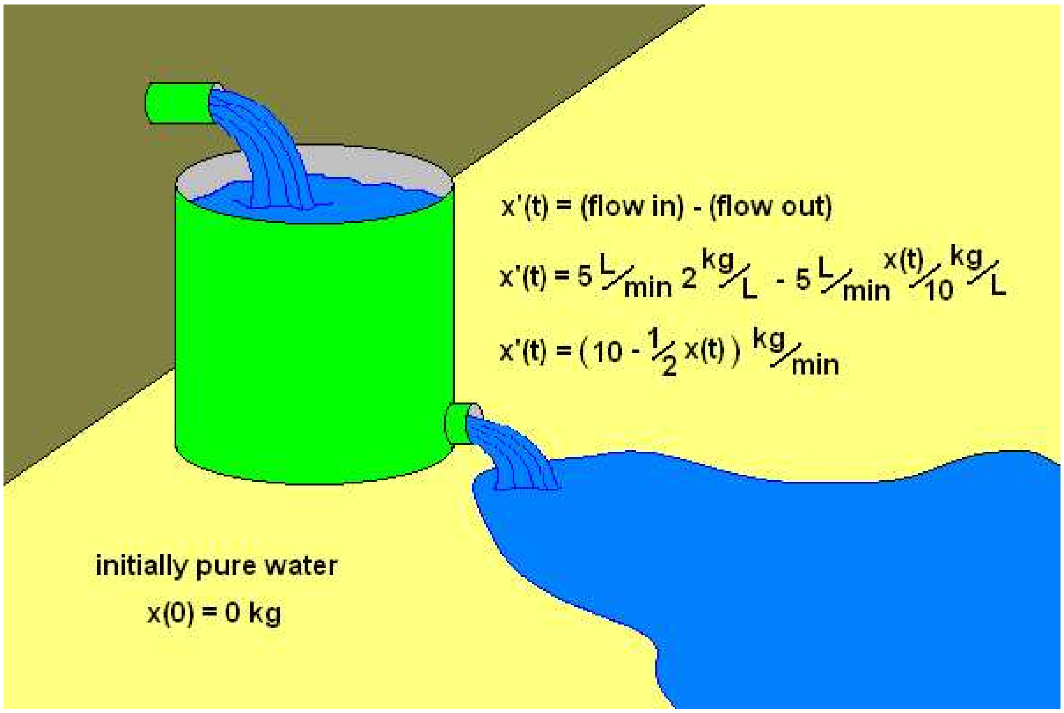

Figure 1: the mixing tank

Since brine flows in at 5 liters per minute and has a concentration of 2 kilograms per liter, we know that salt is flowing into the tank at a rate of 10 kilograms per minute. Brine is flowing out at a rate of 5 liters per minute (keeping a constant 10 liters of brine in the tank at all times). If $x(t)$ is the number of kilograms of salt in the tank after t minutes has elapsed, then the concentration of salt in the tank after t minutes is $x(t)$ divided by 10.

The tank is initially filled with pure water. This tells us that $x(0) = 0$.

Answer (a): We have the following initial value problem:

$$x'(t) = 10 - \frac{1}{2}x(t), \qquad x(0) = 0.$$

(b) Draw the phase line for the differential equation found in part (a).

This is an autonomous first order differential equation so it makes sense to ask you to draw the phase line. In order to draw the phase line, we must first determine all of the equilibrium solutions. We must solve: $x'(t) = 0$. Thus $10 - 0.5x(t) = 0$ so that $x(t) = 20$. Notice that when $x < 20$ that $x' > 0$ (i.e. try $x = 0$ and get $x' = 10$) and when $x > 20$ that $x' < 0$ (i.e. try $x = 40$ and get $x' = 10 - 0.5(40) = -10$). So we have the following phase line:

Answer (b):

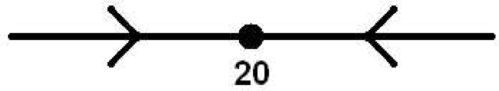

Figure 2: the phase line for $x'(t) = 10 - 0.5x(t)$

(c) What is $\lim_{t\to\infty} x(t)$?

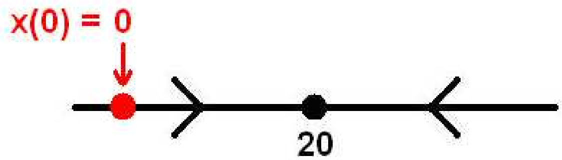

Figure 3: the phase line for $x'(t) = 10 - 0.5x(t)$ with initial condition $x = 0$

The arrow on the phase line points from 0 to 20.

Answer (c): Thus we have that $\lim_{t\to\infty} x(t) = 20$.

#3 (32 points) Solve the following initial value problems:
(You do not need to make your solutions explicit. Implicit solutions are fine.)

(a) $$y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sec(y^2)}{x}, \qquad y(1) = 0$$

This is a separable equation. After multiplying both sides by $\mathrm{d}x/\sec(y^2)$, we get:

$$\frac{y}{\sec(y^2)}\,\mathrm{d}y = \frac{1}{x}\,\mathrm{d}x.$$

Then using the fact that $1/\sec(u) = \cos(u)$ we get:

$$y\cos(y^2)\,\mathrm{d}y = \frac{1}{x}\,\mathrm{d}x.$$

Next, using the substitution $u = y^2$ and $\mathrm{d}u = 2y\,\mathrm{d}y$, we get:

$$\int y\cos(y^2)\,\mathrm{d}y = \int \frac{1}{2}\cos(u)\,\mathrm{d}u = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin(y^2) + C.$$

Therefore, $\frac{1}{2}\sin(y^2) = \ln|x| + C$. Substitute the initial condition into the equation to find that $\frac{1}{2}\sin(0^2) = \ln(1) + C$ which is $C = 0$. Finally, multiply both sides by 2 and get that $\sin(y^2) = 2\ln|x| = \ln(x^2)$.

Answer (a): $\sin(y^2) = \ln(x^2)$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{x}=e^{-x}, \qquad y(1)=e^{-1}$

This is a first order linear equation, so we can use our formula with the integrating factor μ. Notice that $P(x)=\frac{1}{x}$ and $Q(x)=e^{-x}$.

$$\mu(x)=e^{\int P(x)\,\mathrm{d}x}=e^{\int \frac{1}{x}\,\mathrm{d}x}=e^{\ln|x|}=x$$

Thus the general solution is

$$\begin{aligned} y &= \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)\,\mathrm{d}x+C\right) \\ &= \frac{1}{x}\left(\int xe^{-x}\,\mathrm{d}x+C\right) \\ &= \frac{1}{x}\left(-xe^{-x}-\int -e^{-x}\,\mathrm{d}x+C\right) \\ &= \frac{1}{x}\left((-1-x)e^{-x}+C\right). \end{aligned}$$

We have the initial condition $y(1)=e^{-1}$. Thus $e^{-1}=\frac{1}{1}\left((-1-1)e^{-1}+C\right)=-2e^{-1}+C$. This tells us that $C=3e^{-1}$.

Answer (b):

$$y=-\frac{e^{-x}}{x}-e^{-x}+\frac{3e^{-1}}{x}$$

(c) $\left(3x^2+2xy^2\right)\mathrm{d}x+\left(2x^2y+3\right)\mathrm{d}y=0, \qquad y(0)=2$

A quick check of partial derivatives tells us that this equation is exact. We have that

$$\frac{\partial}{\partial y}\left(3x^2+2xy^2\right)=4xy$$

and

$$\frac{\partial}{\partial x}\left(3+2x^2y\right)=4xy$$

are equal. Next, we integrate both terms:

$$\int\left(3x^2+2xy^2\right)\mathrm{d}x=x^3+x^2y^2+(y \text{ terms only})$$

and

$$\int\left(2x^2y+3\right)\mathrm{d}y=x^2y^2+3y+(x \text{ terms only}).$$

Comparing terms, $3y$ is the missing term in the first equation and x^3 is the missing term is the second equation. Thus the general solution is $x^3+x^2y^2+3y=C$.

Plugging in the initial condition $y(0) = 2$ we get that $0^3 + 2^2 0^2 + 3(2) = 6 = C$.

Answer (c): $x^3 + x^2y^2 + 3y = 6$

(d) $y'' - y' - 2y = 0, \qquad y(0) = 2$ and $y'(0) = 1$

This is a homogeneous linear equation with constant coefficients, so we write down the characteristic equation: $r^2 - r - 2 = 0$. This factors as follows: $(r-2)(r+1) = 0$. The solutions are 2 and -1. Thus the general solution is $y = C_1e^{2x} + C_2e^{-x}$.

To find C_1 and C_2 we must compute y' which is $y' = 2C_1e^{2x} - C_2e^{-x}$. Now plugging in the initial conditions we get: $2 = C_1 + C_2$ and $1 = 2C_1 - C_2$. Adding these equations together we find that $3 = 3C_1$. Thus $C_1 = 1$ and $C_2 = 1$.

Answer (d): $y = e^{2x} + e^{-x}$

#4 (30 points) Find general solutions for the following equations:

(a) $y'' + 6y' + 13y = 13x + 19$

This is a second order nonhomogeneous linear differential equation with constant coefficients. First, we must solve the corresponding homogeneous equation. The characteristic equation is $r^2 + 6r + 13 = 0$ which factors to $(r - 3 - 2i)(r - 3 + 2i) = 0$ (just use the quadratic formula). Thus $\alpha = -3$ and $\beta = 2$, so we have $y_h = C_1e^{-3x}\sin(2x) + C_2e^{-3x}\cos(2x)$.

Now the nonhomogeneous term is made up of parts of solutions to linear differential equations with constant coefficients, so we can use the method of undetermined coefficients. Our guess for a particular solution is $y_p = Ax + B$. Notice that this does not overlap any part of y_h, so no adjustment is needed.

Next we compute $y_p' = A$ and $y_p'' = 0$. Plugging this into the equation we get: $0 + 6A + 13(Ax + B) = 13x + 19$. Thus we have that $13Ax + (6A + 13B) = 13x + 19$ so that $13A = 13$ and $6A + 13B = 19$. Therefore, $A = B = 1$ so that $y_p = x + 1$.

The general solution is homogeneous solution plus a particular solution $y = y_h + y_p$.

Answer (a): $y = C_1e^{-3x}\sin(2x) + C_2e^{-3x}\cos(2x) + x + 1$

(b) $y'' + 2y' + y = 2e^{-t}$

This is a second order nonhomogeneous linear differential equation with constant coefficients. First, we must solve the corresponding homogeneous equation. The characteristic equation is $r^2 + 2r + 1 = 0$ which factors to $(r+1)^2 = 0$. So we have that $y_h = C_1e^{-t} + C_2te^{-t}$.

Now the nonhomogeneous term is made up of parts of solutions to linear differential equations with constant coefficients, so we can use the method of undetermined coefficients. Our guess would normally be Ae^{-t}, but this is part of y_h. Multiplying by t is not enough because Ate^{-t} is also part of y_h. Thus our guess is $y_p = At^2e^{-t}$.

Next we compute $y_p' = 2Ate^{-t} + At^2(-e^{-t}) = A(2t-t^2)e^{-t}$ and $y_p'' = A(2-2t)e^{-t} + A(2t-t^2)(-e^{-t}) = A(t^2-4t+2)e^{-t}$. Plugging this into the equation we get:

$$A(t^2-4t+2)e^{-t} + 2A(2t-t^2)e^{-t} + At^2e^{-t} = 2e^{-t}.$$

After cleaning this up, we find that $2Ae^{-t} = 2e^{-t}$ and thus $A = 1$ so that $y_p = t^2e^{-t}$.
Several students decided to find a particular solution using variation of parameteres. Let us compute y_p this way also. First we assign $y_1 = e^{-t}$ and $y_2 = te^{-t}$. Let us compute $y_1y_2' - y_1'y_2 = e^{-t}((1-t)e^{-t}) - (-e^{-t})te^{-t} = e^{-2t} - te^{-2t} + te^{-2t} = e^{-2t}$. Using our formulas we have that:

$$v_1(t) = \int \frac{-g(t)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}\,\mathrm{d}t = \int \frac{-2e^{-t}te^{-t}}{e^{-2t}}\,\mathrm{d}t = \int -2t\,\mathrm{d}t = -t^2$$

and

$$v_2(t) = \int \frac{g(t)y_1(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}\,\mathrm{d}t = \int \frac{2e^{-t}e^{-t}}{e^{-2t}}\,\mathrm{d}t = \int 2\,\mathrm{d}t = 2t.$$

Thus $y_p = v_1y_1 + v_2y_2 = -t^2e^{-t} + 2t(te^{-t}) = t^2e^{-t}$ (just as before).
The general solution is homogeneous solution plus a particular solution $y = y_h + y_p$.

Answer (b): $y = C_1e^{-t} + C_2te^{-t} + t^2e^{-t}$

(c) $y'' + y = \csc(x)$

This is a second order nonhomogeneous linear differential equation with constant coefficients. First, we must solve the corresponding homogeneous equation. The characteristic equation is $r^2 + 1 = 0$ whose solutions are $r = \pm i$. Thus $y_h = C_1\sin(x) + C_2\cos(x)$.
Now this time the nonhomogeneous term is not made up of parts of solutions to linear differential equations with constant coefficients, so we cannot use the method of undetermined coefficients. We must use variation of parameters.
First, we assign $y_1 = \sin(x)$ and $y_2 = \cos(x)$ and compute $y_1y_2' - y_1'y_2 = \sin(x)(-\sin(x)) - \cos(x)\cos(x) = -(\sin^2(x) + \cos^2(x)) = -1$. Using our formulas we have that:

$$\begin{aligned} v_1(x) = \int \frac{-g(x)y_2(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}\,\mathrm{d}x &= \int \frac{-\csc(x)\cos(x)}{-1}\,\mathrm{d}x \\ &= \int \frac{\cos(x)}{\sin(x)}\,\mathrm{d}x = \ln|\sin(x)| \end{aligned}$$

(to see this last step consider the substitution $u = \sin(x)$ and $\mathrm{d}u = \cos(x)\mathrm{d}x$) and

$$\begin{aligned} v_2(x) = \int \frac{g(x)y_1(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}\,\mathrm{d}x &= \int \frac{\csc(x)\sin(x)}{-1}\,\mathrm{d}x \\ &= \int -\mathrm{d}x = -x. \end{aligned}$$

Thus $y_p = v_1y_1 + v_2y_2 = \sin(x)\ln|\sin(x)| - x\cos(x)$.
The general solution is homogeneous solution plus a particular solution $y = y_h + y_p$.

Answer (c): $y = C_1\sin(x) + C_2\cos(x) + \sin(x)\ln|\sin(x)| - x\cos(x)$

#5 (10 points) Find linear differential operators which annihilate the following functions:

(a) $6xe^{2x} + 5$

We know that $(D-2)$ annihilates e^{2x} and thus $(D-2)^2$ will annihilate xe^{2x}. The constant 5 is annihilated by D.

Answer (a): The function is annihilated by $(D-2)^2D$.

(b) $2e^{-x}\sin(3x) + 15e^{-x}\cos(3x) + 7e^{4x}$

The term $e^{-x}\sin(3x)$ tells us to consider $\alpha = -1$ and $\beta = 3$ and thus $(D-(-1)-3i)(D-(-1)+3i) = D^2 + 2D + 10$. The operator $D^2 + 2D + 10$ will also annihilate $e^{-x}\cos(3x)$. Finally, notice that e^{4x} is annihilated by $(D-4)$.

Answer (b): The function is annihilated by $(D^2 + 2D + 10)(D-4)$.