Ma341-004: Test #2
Answer Key

Friday, June 14, 2005
Instructor: Dr. Bill Cook

#1 (24 points) Compute the Laplace transform, $\mathcal{L}\{f(t)\}(s)$, for each of the following:

(a) $f(t) = \dfrac{1}{\sqrt{t}} + te^{-2t} - 1$ $\left(\textit{Hint: } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}\right)$

(b) $f(t) = e^{2t}\cos(3t) - 5e^{-t}\sin(2t)$

(c) $f(t) = t\sin(t)$

(d) $f(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ t & 1 < t \leq 2 \\ e^t + 2 & 2 < t \end{cases}$

$$\text{(a)} \quad f(t) = \frac{1}{\sqrt{t}} + te^{-2t} - 1 = t^{-\frac{1}{2}} + e^{-2t} \cdot t - 1$$

Using formulae (#19, #16, and #14) we have that:

Answer (a):

$$\mathcal{L}\{f(t)\}(s) = \sqrt{\frac{\pi}{s}} + \frac{1}{(s+2)^2} - \frac{1}{s}$$

$$\text{(b)} \quad f(t) = e^{2t}\cos(3t) - 5e^{-t}\sin(2t)$$

Using formulae (#25 and #26) we have that:

Answer (b):

$$\mathcal{L}\{f(t)\}(s) = \frac{s-2}{(s-2)^2+9} - \frac{10}{(s+1)^2+4}$$

$$\text{(c)} \quad f(t) = t\sin(t)$$

Use formula #30 or...

Using formula #5 we have that:

$$\mathcal{L}\{t\sin(t)\}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\left[\mathcal{L}\{\sin(t)\}(s)\right] = -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{s^2+1}\right]$$

Answer (c):

$$\mathcal{L}\{t\sin(t)\}(s) = \frac{2s}{(s^2+1)^2}$$

(d) $f(t) = 1 + u(t-1)\,[t-1] + u(t-2)\,\left[e^t + 2 - t\right] = 1 + (t-1)u(t-1) + \left(e^2 \cdot e^{t-2} - (t-2)\right)u(t-2)$

Now we use formula #10 to find that:

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{s} + e^{-s}\mathcal{L}\{t\}(s) + e^{-2s}\mathcal{L}\left\{e^2 e^t - t\right\}(s) = \frac{1}{s} + \frac{e^{-s}}{s^2} + e^{-2s}\left(e^2\frac{1}{s-1} - \frac{1}{s^2}\right)$$

Answer (d):

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{s} + \frac{e^{-s}}{s^2} + \frac{e^{-2(s-2)}}{s-1} - \frac{e^{-2s}}{s^2}$$

#2 (32 points) Compute the inverse Laplace transform, $\mathcal{L}^{-1}\{F(s)\}(t)$, for each of the following:

(a) $F(s) = \dfrac{s+1}{s^2-2s+5}$ (b) $F(s) = \dfrac{2s^2}{(s^2+1)(s-1)}$

(c) $F(s) = \dfrac{e^{-s}(4s+2)}{s(s+1)}$ (d) $F(s) = \dfrac{12s}{(s^2+4)(s^2+9)(s-1)s^4}$

Note: For part (d), please give your answer in terms of a (multi-)convolution product

$$\text{(a)} \quad F(s) = \frac{s+1}{s^2-2s+5} = \frac{s+1}{(s-1)^2+4}$$

We want to use formulae (#25 and #26) so we complete the square. Then we rewrite the numerator in terms of "$(s-1)$" and get:

$$F(s) = \frac{(s-1)+2}{(s-1)^2+4} = \frac{(s-1)}{(s-1)^2+4} + \frac{2}{(s-1)^2+4}$$

Answer (a): $\mathcal{L}^{-1}\{F(s)\}(t) = e^t\cos(2t) + e^t\sin(2t)$

$$\text{(b)} \quad F(s) = \frac{2s^2}{(s^2+1)(s-1)}$$

First we must preform the partial fraction decomposition.

$$\frac{2s^2+3s-1}{(s^2+1)(s-1)} = \frac{As+B}{s^2+1} + \frac{C}{s-1}$$

Clearing the denominators, we get: $2s^2 = (As+B)(s-1) + C(s^2+1) = As^2 - As + Bs - B + Cs^2 + C$. Simplifying we get: $2s^2 = (A+B)s^2 + (-A+B)s + (-B+C)$. This gives us the equations $A+B=2$, $-A+B=0$, and $-B+C=0$. Thus $A=B=C$ and thus $2A=2$. Hence $A=B=C=1$. Therefore,

$$F(s) = \frac{s+1}{s^2+1} + \frac{1}{s-1} = \frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{s-1}$$

Answer (b): $\mathcal{L}^{-1}\{F(s)\}(t) = \cos(t) + \sin(t) + e^t$

$$\text{(c)} \quad F(s) = \frac{e^{-s}(4s+2)}{s(s+1)}$$

First we must figure out the inverse Laplace transform for $F(s)$ without the e^{-s} term. After that, we can use formula #10.

$$\frac{4s+2}{s(s+1)} = \frac{A}{s} + \frac{B}{(s+1)}$$

Clearing the denominators, we get: $4s+2 = A(s+1) + Bs = (A+B)s + A$. Thus $A+B=4$ and $A=2$. Therefore, $A=B=2$ and thus

$$\frac{4s+2}{s(s+1)} = \frac{2}{s} + \frac{2}{(s+1)}$$

Whose inverse transform is $2+2e^{-t}$. Thus, in formula #10, $f(t-1) = 2+2e^{-(t-1)}$.

Answer (c):

$$\mathcal{L}^{-1}\{F(s)\}(t) = \left(2+2e^{-(t-1)}\right) u(t-1)$$

$$\text{(d)} \quad F(s) = \frac{12s}{(s^2+4)(s^2+9)(s-1)s^4}$$

To find the answer (without convolutions) would require a huge and horrible partial fractions problem. But (allowing convolutions) we use formula #8.

$$F(s) = \frac{2}{s^2+4} \cdot \frac{s}{s^2+9} \cdot \frac{1}{s-1} \cdot \frac{3!}{s^4}$$

Answer (d):

$$\mathcal{L}^{-1}\{F(s)\}(t) = \sin(2t) * \cos(3t) * e^t * t^3$$

Note: There are many possible answers to part (d). All of them represent the same function, but look very different. For example, we could have cancelled out s and got:

$$F(s) = \frac{2}{s^2+4} \cdot \frac{3}{s^2+9} \cdot \frac{1}{s-1} \cdot \frac{2}{s^3}$$

This given the answer:

$$\mathcal{L}^{-1}\{F(s)\}(t) = \sin(2t) * \sin(3t) * e^t * t^2$$

#3 (30 points) Solve the following initial value problems using Laplace transforms.
(a) $y'' + y = 2e^t$, $y(0) = 1$, and $y'(0) = 2$.

First, we must transform the problem. Note that:

$$\mathcal{L}\{y\}(s) = Y$$

and

$$\mathcal{L}\{y''\}(s) = \mathcal{L}\{y\}(s) \cdot s^2 - (1)s - 2 = Ys^2 - s - 2$$

The right hand side transforms to $\frac{2}{s-1}$.
This gives us the following equation:

$$Ys^2 - s - 2 + Y = \frac{2}{s-1}$$

which is...

$$Y(s^2+1) = \frac{2}{s-1} + s + 2 = \frac{2 + (s+2)(s-1)}{s-1} = \frac{s^2+s}{s-1}$$

Therefore,

$$Y = \frac{s^2+s}{(s-1)(s^2+1)}$$

Next, we must do partial fractions...

$$\frac{s^2+s}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

Clearing the denominators, we get: $s^2 + s = A(s^2+1) + (Bs+C)(s-1) = (A+B)s^2 + (-B+C)s + (A-C)$. This gives us the equations: $A + B = 1$, $-B + C = 1$, and $A - C = 0$. Thus $A = C$ and hence $-B + A = 1$ adding this to the first equation gives, $2A = 2$. Thus $A = C = 1$ and $B = 0$. So we have:

$$Y = \frac{1}{s-1} + \frac{1}{s^2+1}$$

Finally, we do the inverse transform.

Answer (a): $y = e^t + \sin(t)$

(b) $y'' - 2y' + y = u(t-1)$, $y(0) = 0$, and $y'(0) = 1$ ($u(t)$ is the unit step function).

First, we must transform the problem. Note that:

$$\mathcal{L}\{y\}(s) = Y,$$

$$\mathcal{L}\{y'\}(s) = \mathcal{L}\{y\}(s)\cdot s - 0 = Ys,$$

and

$$\mathcal{L}\{y''\}(s) = \mathcal{L}\{y\}(s)\cdot s^2 - (0)s - 1 = Ys^2 - 1$$

The right hand side transforms to $\frac{e^{-s}}{s}$.
This gives us the following equation:

$$(Ys^2 - 1) - 2(Ys) + Y = \frac{e^{-s}}{s}$$

Simplifying we get:

$$Y(s-1)^2 = Y(s^2 - 2s + 1) = \frac{e^{-s}}{s} + 1$$

Therefore,

$$Y = \frac{e^{-s}}{s(s-1)^2} + \frac{1}{(s-1)^2}$$

We must do partial fractions to take care of the first term:

$$\frac{1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2}$$

Clearing the denominators, we get: $1 = A(s-1)^2 + Bs(s-1) + Cs = (A+B)s^2 + (-2A - B + C)s + A$. This gives us the equations: $A + B = 0$, $-2A - B + C = 0$ and $A = 1$. Thus $B = -1$ and hence $-2 + 1 + C = 0$. So $A = C = 1$ and $B = -1$. Therefore,

$$Y = \frac{e^{-s}}{\frac{1}{s} - \frac{1}{(s-1)} + \frac{1}{(s-1)^2}} + \frac{1}{(s-1)^2}$$

The inverse transform of

$$\frac{1}{s} - \frac{1}{(s-1)} + \frac{1}{(s-1)^2}$$

is $1 - e^t + te^t$. Using equation #10 we get:

Answer (b): $y = te^t + (1 - e^{t-1} + (t-1)e^{t-1})\, u(t-1)$

#4 (16 points) Use Laplace transforms to solve the following system of differential equations:

$$\begin{aligned} x' &= x + y \qquad & x(0) = 0 \\ y' &= x + y \qquad & y(0) = 6 \end{aligned}$$

First, we must transform the problem. Note that:

$$\mathcal{L}\{x\}(s) = X,$$

$$\mathcal{L}\{x'\}(s) = \mathcal{L}\{x\}(s)\cdot s - 0 = Xs,$$

$$\mathcal{L}\{y\}(s) = Y,$$

and

$$\mathcal{L}\{y'\}(s) = \mathcal{L}\{y\}(s) \cdot s - 6 = Ys - 6$$

Thus we have the following equations: $Xs = X + Y$ and $Ys - 6 = X + Y$. Thus $Xs = Ys - 6$. This tells us that:

$$Y = X + \frac{6}{s}$$

Translating back, this means that $y(t) = x(t) + 6$.

Next, let us eliminate Y by substituting $Y = X + \frac{6}{s}$ into $Xs = X + Y$. This gives us:

$$Xs = X + X + \frac{6}{s}$$

which is...

$$X(s-2) = \frac{6}{s}$$

Thus...

$$X = \frac{6}{s(s-2)}$$

Next, we do partial fracions:

$$\frac{6}{s(s-2)} = \frac{A}{s} + \frac{B}{(s-2)}$$

Clearing the denomiators, we get: $6 = A(s-2) + Bs = (A+B)s - 2A$. Thus $A + B = 0$ and $-2A = 6$. So that $A = -3$ and $B = 3$. Therefore,

$$X = \frac{-3}{s} + \frac{3}{(s-2)}$$

Thus, $x(t) = -3 + 3e^{2t}$.

Answer: $x(t) = -3 + 3e^{2t}$ and $y(t) = 3 + 3e^{2t}$