Ma341-004: Test #3
Answer Key

Friday, June 22, 2005
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#1 (15 points) Consider the following system of differential equations:

$$\begin{aligned} y'''(t) + 2y'(t) - x(t) &= 6 \\ x''(t) + y''(t) - 2x'(t) &= \sin(t) \end{aligned}$$

(a) Convert this system into an equivalent system of first order differential equations.

We need to reduce the order of both equations. To do this notice that y's highest order derivative is y''' and x's highest order derivative is x''. This we introduce the following new variables:

$$\begin{aligned} x_1 &= x \\ x_2 &= x' \\ \\ x_3 &= y \\ x_4 &= y' \\ x_5 &= y'' \end{aligned}$$

This introduces the equations:

$$\begin{aligned} x_1' &= x_2 \\ \\ x_3' &= x_4 \\ x_4' &= x_5 \end{aligned}$$

The old equations now read:

$$\begin{aligned} x_5' + 2x_4 - x_1 &= 6 \\ x_2' + x_5 - 2x_2 &= \sin(t) \end{aligned}$$

Rewriting all of these gives us our answer.

Answer (a):

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= 2x_2 - x_5 + \sin(t) \\ x_3' &= x_4 \\ x_4' &= x_5 \\ x_5' &= x_1 - 2x_4 + 6 \end{aligned}$$

(b) Rewrite your answer to part (a) in matrix normal form.

Answer (b):

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ \sin(t) \\ 0 \\ 0 \\ 6 \end{bmatrix}$$

#2 (20 points) Find the general solution for the following system of differential equations:

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{x}(t).$$

We need to start by finding the eigenvalues of the coefficient matrix. Since the matrix is upper-triangular we can just read them off of the diagonal (i.e. 1, 2, and 3). If you aren't aware of this "trick," you can compute the determinate of $A - \lambda I$, set is equal to zero, and find the solutions.

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{bmatrix} 1-\lambda & 4 & 6 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) \\ &= (1-\lambda)\det\left(\begin{bmatrix} 2-\lambda & 5 \\ 0 & 3-\lambda \end{bmatrix}\right) - 0 + 0 \\ &= (1-\lambda)(2-\lambda)(3-\lambda) = 0 \end{aligned}$$

Thus $\lambda = 1$, 2, or 3.

Now we need to find 1 eigenvector for each eigenvalue (since each eigenvalue has multiplicity 1). We will start with $\lambda = 1$.

We must find a non-zero solution to the system of equations $(A - I)u = 0$.

$$\begin{bmatrix} 1-1 & 4 & 6 & \vdots & 0 \\ 0 & 2-1 & 5 & \vdots & 0 \\ 0 & 0 & 3-1 & \vdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 6 & \vdots & 0 \\ 0 & 1 & 5 & \vdots & 0 \\ 0 & 0 & 2 & \vdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 5 & \vdots & 0 \\ 0 & 4 & 6 & \vdots & 0 \\ 0 & 0 & 2 & \vdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 5 & \vdots & 0 \\ 0 & 0 & 6 & \vdots & 0 \\ 0 & 0 & 2 & \vdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

Thus if

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix},$$

then $u_2 = 0$ and $u_3 = 0$, but u_1 is a free parameter – say $u_1 = s$. Therefore,

$$u = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix}.$$

For simplicity, choose $s = 1$ and get:

$$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Next, we need to find an eigenvector for the eigenvalue $\lambda = 2$. That is, a non-zero solution to the equations $(A - 2I)u = 0$.

$$\begin{bmatrix} 1-2 & 4 & 6 & \vdots & 0 \\ 0 & 2-2 & 5 & \vdots & 0 \\ 0 & 0 & 3-2 & \vdots & 0 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 6 & \vdots & 0 \\ 0 & 0 & 5 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 & -6 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -4 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

Thus $u_1 - 4u_2 = 0$. u_2 is a free parameter – say $u_2 = s$. So that $u_1 = 4s$, $u_2 = s$, and $u_3 = 0$. We get:

$$u = \begin{bmatrix} 4s \\ s \\ 0 \end{bmatrix}.$$

For simplicity, choose $s = 1$ and get:

$$u = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}.$$

Finally, we need to find an eigenvector for the eigenvalue $\lambda = 3$. That is, a non-zero solution to the equations $(A - 3I)u = 0$.

$$\begin{bmatrix} 1-3 & 4 & 6 & \vdots & 0 \\ 0 & 2-3 & 5 & \vdots & 0 \\ 0 & 0 & 3-3 & \vdots & 0 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 6 & \vdots & 0 \\ 0 & -1 & 5 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 26 & \vdots & 0 \\ 0 & -1 & 5 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & -13 & \vdots & 0 \\ 0 & 1 & -5 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

Thus $u_1 - 13u_3 = 0$, $u_2 - 5u_3 = 0$, and u_3 is a free parameter – say $u_3 = s$. Thus $u_1 = 13s$, $u_2 = 5s$, and $u_3 = s$. We get:

$$u = \begin{bmatrix} 13s \\ 5s \\ s \end{bmatrix}.$$

For simplicity, choose $s = 1$ and get:

$$u = \begin{bmatrix} 13 \\ 5 \\ 1 \end{bmatrix}.$$

Thus the first eigenvalue, eigenvector pair gives us the solution:

$$e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The second eigenvalue, eigenvector pair gives us the solution:

$$e^{2t} \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}.$$

And the third eigenvalue, eigenvector pair gives us the solution:

$$e^{3t} \begin{bmatrix} 13 \\ 5 \\ 1 \end{bmatrix}.$$

Answer: The general solution is...

$$\mathbf{x}(t) = C_1 e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + C_3 e^{3t} \begin{bmatrix} 13 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} e^t & 4e^{2t} & 13e^{3t} \\ 0 & e^{2t} & 5e^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

#3 (20 points) Solve the following initial value problem:

$$\mathbf{x}'(t) = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{x}(t) \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Please note:

The matrix $\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ has an eigenvalue $-1 + i$ with corresponding eigenvector $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

We know that complex eigenvalues of real matrices come in conjugate pairs. Thus the other eigenvalue for this matrix must be $-1-i$ and its eigenvector must be:

$$\overline{\begin{bmatrix} i \\ 1 \end{bmatrix}} = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

But this does not really matter because we just want to find 2 linearly independant solutions. According to our formula, if we have an eigenvalue $\alpha + \beta i$ with eigenvector $u = a + bi$ then...

$$\mathbf{x}_1(t) = e^{\alpha t}\cos(\beta t)a - e^{\alpha t}\sin(\beta t)b$$

and

$$\mathbf{x}_2(t) = e^{\alpha t}\sin(\beta t)a + e^{\alpha t}\cos(\beta t)b$$

are solutions. In our problem

$$a = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \alpha = -1, \qquad \text{and} \qquad \beta = 1.$$

Thus we have that

$$\mathbf{x}_1(t) = e^{-t}\cos(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - e^{-t}\sin(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -e^{-t}\sin(t) \\ e^{-t}\cos(t) \end{bmatrix}$$

and

$$\mathbf{x}_2(t) = e^{-t}\sin(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + e^{-t}\cos(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-t}\cos(t) \\ e^{-t}\sin(t) \end{bmatrix}$$

are our solutions. Therefore, the general solution is

$$\mathbf{x}(t) = C_1\begin{bmatrix} -e^{-t}\sin(t) \\ e^{-t}\cos(t) \end{bmatrix} + C_2\begin{bmatrix} e^{-t}\cos(t) \\ e^{-t}\sin(t) \end{bmatrix}.$$

Now we need to plug in the initial conditions and solve for C_1 and C_2. We get:

$$\begin{aligned} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathbf{x}(0) &= C_1\begin{bmatrix} -e^{-0}\sin(0) \\ e^{-0}\cos(0) \end{bmatrix} + C_2\begin{bmatrix} e^{-0}\cos(0) \\ e^{-0}\sin(0) \end{bmatrix} \\ &= C_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} C_2 \\ C_1 \end{bmatrix} \end{aligned}$$

So we have that $C_1 = 2$ and $C_2 = 1$.

Answer:

$$\mathbf{x}(t) = \begin{bmatrix} -2e^{-t}\sin(t) + e^{-t}\cos(t) \\ 2e^{-t}\cos(t) + e^{-t}\sin(t) \end{bmatrix}$$

#4 (20 points) Compute the matrix exponential e^{At} where

$$A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}.$$

Let's compute the matrix exponential the standard way first, then I'll use a *major* shortcut.

To compute the matrix exponential we need a *basis* of generalized eigenvectors. Thus our first step must be finding the eigenvalues.

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} -3-\lambda & 2 \\ -2 & 1-\lambda \end{bmatrix}\right) = (-3-\lambda)(1-\lambda)-(-2)2 = \lambda^2+2\lambda-3+4 = (\lambda+1)^2 = 0$$

So we have that $\lambda = -1$ is an eigenvalue (with multiplicity 2).

We need to find two (linearly independent) eigenvectors for the eigenvalue $\lambda = -1$ since it has multiplicity 2. So we solve the equations $(A - (-1)I)u = 0$ that is $(A + I)u = 0$.

$$\left[\begin{array}{cc|c} -3+1 & 2 & 0 \\ -2 & 1+1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2 & 2 & 0 \\ -2 & 2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus $u_1 - u_2 = 0$ and u_2 is a free parameter – say $u_2 = s$. So that $u_1 = u_2 = s$. We get:

$$u = \begin{bmatrix} s \\ s \end{bmatrix} \qquad \text{for simplicity take } s = 1 \text{ and get} \qquad u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Next, notice that we *needed* two eigenvectors, but we only had one free parameter. Thus we must resort to finding *generalized eigenvectors*. We solved $(A+I)u = 0$ so next we must solve $(A + I)^2u = 0$.

$$(A+I)^2 = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus we must solve the system $0u = 0$. In this case both u_1 and u_2 are free parameters. So we can choose any vector we want – *as long as it isn't a multiple of our first vector!* For simplicity, choose $u_1 = 1$ and $u_2 = 0$. That is

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The first eigenvector gives us the solution:

$$e^{\lambda t}u = e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The second vector is a generalized eigenvector, so it gives us the solution:

$$e^{\lambda t}\left(u + (A+I)ut\right) = e^{-t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}t\right) = e^{-t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix}t\right) = \begin{bmatrix} e^{-t} - 2te^{-t} \\ -2te^{-t} \end{bmatrix}.$$

Thus the system $\mathbf{x}' = A\mathbf{x}$ has a fundamental matrix:

$$X(t) = \begin{bmatrix} e^{-t} & e^{-t} - 2te^{-t} \\ e^{-t} & -2te^{-t} \end{bmatrix}.$$

Then we have that

$$X(0) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

This means that

$$X(0)^{-1} = \frac{1}{(1)0 - (1)(1)} \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

Therefore,

$$e^{At} = X(t)X(0)^{-1} = \begin{bmatrix} e^{-t} & e^{-t} - 2te^{-t} \\ e^{-t} & -2te^{-t} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} e^{-t} - 2te^{-t} & 2te^{-t} \\ -2te^{-t} & e^{-t} + 2te^{-t} \end{bmatrix}.$$

Major shortcut: We know that (for *any* λ),

$$e^{At} = e^{\lambda t} e^{(A-\lambda I)t} = e^{\lambda t}\left(I + (A - \lambda I)t + \frac{(A-\lambda I)^2}{2!}t^2 + ...\right)$$

Consider the eigenvalue $\lambda = -1$ (our *only* eigenvalue).

$$e^{At} = e^{-t}\left(I + (A + I)t + \frac{(A+I)^2}{2!}t^2 + ...\right)$$

So we compute I, $(A+I)$, $(A+I)^2$,...

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A + I = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}, \qquad (A+I)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad (A+I)^3 = 0, ...$$

Therefore,

$$e^{At} = e^{-t}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} t + 0 + 0 + ...\right) = e^{-t}\begin{bmatrix} 1 - 2t & 2t \\ -2t & 1 + 2t \end{bmatrix}$$

Answer:

$$e^{At} = \begin{bmatrix} e^{-t} - 2te^{-t} & 2te^{-t} \\ -2te^{-t} & e^{-t} + 2te^{-t} \end{bmatrix}$$

#5 (25 points) Find the general solution for the following system of differential equations:

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ e^t \end{bmatrix}.$$

Please note:

$$X(t)=\begin{bmatrix}1 & 2e^t\\ 0 & e^t\end{bmatrix} \text{ is a fundamental matrix for } \mathbf{x}'(t)=\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix}\mathbf{x}(t).$$

We have been given a fundamental matrix for the corresponding homogeneous system. The method of variation of parameters then gives us a particular solution to our non-homogeneous system.

$$\mathbf{x}_p(t)=X(t)\int X(t)^{-1}f(t)\mathrm{d}t$$

So we must compute $X(t)^{-1}$. We could either use the formula for the inverse of a 2 by 2 matrix or use row-reduction. Just for the ?fun? of it, I will do row-reduction.

$$\begin{bmatrix}X(t) & \vdots & I\end{bmatrix}=\begin{bmatrix}1 & 2e^t & \vdots & 1 & 0\\ 0 & e^t & \vdots & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & \vdots & 1 & -2\\ 0 & e^t & \vdots & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & \vdots & 1 & -2\\ 0 & 1 & \vdots & 0 & e^{-t}\end{bmatrix}=\begin{bmatrix}I & \vdots & X(t)^{-1}\end{bmatrix}$$

Thus,

$$X(t)^{-1}f(t)=\begin{bmatrix}1 & -2\\ 0 & e^{-t}\end{bmatrix}\begin{bmatrix}1\\ e^t\end{bmatrix}=\begin{bmatrix}1-2e^t\\ 1\end{bmatrix}$$

$$\int X(t)^{-1}f(t)\mathrm{d}t=\int\begin{bmatrix}1-2e^t\\ 1\end{bmatrix}\mathrm{d}t=\begin{bmatrix}t-2e^t\\ t\end{bmatrix}$$

$$\mathbf{x}_p(t)=X(t)\int X(t)^{-1}f(t)\mathrm{d}t=\begin{bmatrix}1 & 2e^t\\ 0 & e^t\end{bmatrix}\begin{bmatrix}t-2e^t\\ t\end{bmatrix}=\begin{bmatrix}t-2e^t+2te^t\\ te^t\end{bmatrix}$$

The general solution is the general solution to the homogeneous system plus a particular solution: $\mathbf{x}(t)=X(t)c+\mathbf{x}_p(t)$.

Answer: C_1 and C_2 are arbitrary constants.

$$\mathbf{x}(t)=\begin{bmatrix}1 & 2e^t\\ 0 & e^t\end{bmatrix}\begin{bmatrix}C_1\\ C_2\end{bmatrix}+\begin{bmatrix}t-2e^t+2te^t\\ te^t\end{bmatrix}=\begin{bmatrix}C_1+t+2(C_2-1)e^t+2te^t\\ C_2e^t+te^t\end{bmatrix}$$