Introduction to Differential Equations: Homework on Laplace Transforms

Directions: Show your work, if you doubt that you've shown enough detail then ask. My instructions supercede the intstructions in the text. This homework will not be collected. A solution will be posted before the test. There is considerable overlap with the practice test. I believe it will take some time to complete all of these, you may not need to do everything to get the idea, some of the difficult problems are too hard for the test but completing them might help you gain skill which is useful beyond this course and my test.

1. Derive trigonometric identities that would allow you to take the Laplace transform of the following functions.
   (a.) $f(t) = \sin(t)\sin(3t)$

   (b.) $f(t) = \cos(3t)\sin(7t)$

   (c.) $f(t) = \sin(t)\cos(t)$

   (d.) $f(t) = \cos^2(3t)$

   (e.) $f(t) = \cos(t+4)$

2. Calculate the Laplace transforms of the following functions using the table of basic Laplace transforms plus possibly the given Theorems. Derive any needed trigonometric identities via the imaginary exponential tricks.
   (a.) $f(t) = \sin(t)\cos(2t)$

   (b.) $f(t) = \sin^2(3t)$

   (c.) $f(t) = \sin^2(t+3)$

   (d.) $f(t) = \sin^2(t+3)\cos(2t)$

   (e.) $f(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ t^2 & 1 \leq t \leq 2 \\ 4\sin(t) & t > 2 \end{cases}.$

   (f.) $f(t) = \begin{cases} t, & 0 \leq t \leq 2 \\ \sin(t) & t > 2 \end{cases}.$

   (g.) $f(t) = te^{-2t} + t\sin(t)$

   (h.) $f(t)$ periodic with $f_T(t) = t$ for $0 \leq t \leq 2$ with $T = 2$

3. Compute the inverse Laplace transforms of,

(a.)

$$F(s) = \frac{3s+9}{s^2-8s+7}$$

(b.)

$$G(s) = e^{-3s}\frac{3s+9}{s^2-8s+7}$$

(c.)

$$F(s) = \frac{10s^2+10s+10}{(s^2+4s+5)s}$$

(d.)

$$F(s) = \frac{s(s+2)}{(s^2+25)[(s+2)^2+16](s-1)}$$

(e.)

$$F(s) = \frac{s^2}{s^2+4s+5}$$

(f.)

$$F(s) = e^{-2s}\frac{1}{s(s^2-8s+7)}$$

(g.)

$$F(s) = e^{-4s} + 1 + \frac{1}{s}$$

please state your answer for (d.) in terms of a convolution product, you should not attempt the integrals involved. For parts (a.),(b.),(c.),(e.),(f.),(g.) if you use the convolution method I expect you to explicitly work out the integral (in other words only use convolution on part d.)

4. Solve the following differential equations with the given initial conditions by the method of Laplace transforms.

   (a.) $y'' + y' - 2y = 0$ where $y(0) = 2$ and $y'(0) = 1$

   (b.) $y'' - 2y' + y = \delta(t-2)$ where $y(0) = 1$ and $y'(0) = 0$

   (c.) $y'' - 8y' + 7y = u(t-2)$ where $y(0) = 0$ and $y'(0) = 0$

   (d.) $y'' + y = \delta(t-2)$ where $y(0) = 1$ and $y'(0) = 2$

   (e.) $y'' - 8y' + 7y = u(t-2) + u(t-4)$ where $y(0) = 0$ and $y'(0) = 0$

   (f.) $y'' + 3ty' - 6y = 1$ where $y(0) = 0$ and $y'(0) = 0$

5. (challenge problem) Let us study the problem of a repeated irreducible quadratic. As of now we have no systematic approach to dealing with these, especially the inverse transform. It is of course straight forward to obtain a repeated irreducible quadratic after taking the Laplace transform of something like $t\sin(t)$, the question that is more subtle how to go backwards. So let us state the problem, find the inverse Laplace transform of the following

$$G(s) = \frac{s^3 + As^2 + Bs + C}{[(s-\alpha)^2 + \beta^2]^2}$$

   where A, B, C, α, β are constants. I have assumed we have a proper fraction to begin with, otherwise we would need to do a long division to start.

6. Solve $y'' + y = 3$ using Laplace transforms. I want to illustrate how Laplace transforms can also be applied to problems without initial conditions.

7. Use Laplace transforms to solve the following system of differential equations.

$$tx' = -x + y - e^{2t} + t^3 \qquad y' = 2y \qquad x(0) = 0, y(0) = 1$$

   notice we cannot use chapter 9 methods here because the A for this system contains functions. Our main results held only for A being a constant matrix.