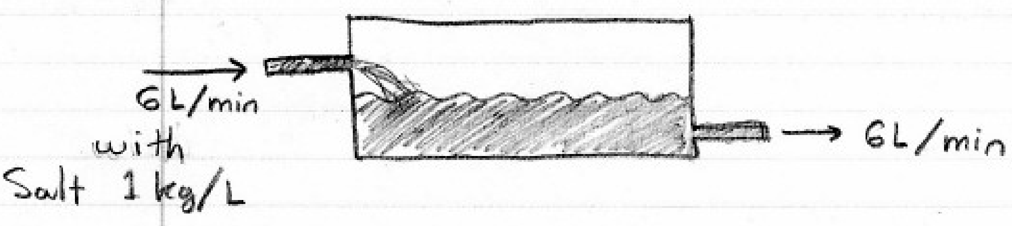
Mixing Tank Problem

Consider the 1000 L tank below.



$X(t)$ = mass of salt in kg at time t in minutes.

The tank is well mixed so that it has a uniform concentration of salt throughout. Assume that intially the tank is full of pure water ; $x(0) = 0$. Find the time t so that the concentration of salt 0.5 kg/L.

Salt enters tank at $\left(6 \frac{L}{min}\right)\left(1 \frac{kg}{L}\right) = 6\ kg/min$

Salt exits tank at $\left(6 \frac{L}{min}\right)\left(\frac{x}{1000L}\right) = \left(\frac{3}{500}\right)x/min$

The amount of salt is modeled by the $DE_{q}{}^{n}$,

$$\frac{dx}{dt} = (\text{rate out}) - (\text{rate in}) = 6 - \frac{3x}{500}$$

We can solve $\frac{dx}{dt} + \frac{3x}{500} = 6$ using integrating factor

$$\mu = \exp\left(\int \frac{3}{500} dt\right) = e^{\frac{3t}{500}}$$

$$e^{\frac{3t}{500}} \frac{dx}{dt} + e^{\frac{3t}{500}} \frac{3}{500} x = \frac{d}{dt}\left(e^{\frac{3t}{500}} x\right) = 6e^{\frac{3t}{500}}$$

$$\Rightarrow e^{-3t/500} x = \frac{500\,(6)}{3} e^{3t/500} + C$$

$$\Rightarrow x = 1000 + Ce^{-3t/500}$$

$x(0) = 0 \Rightarrow C = -1000$

$\therefore \boxed{x = 1000 - 1000e^{-3t/500}}$

Now find t such that

$$\frac{x(t)}{1000} = \frac{1000 - 1000e^{-3t/500}}{1000} = 1 - e^{-3t/500} = 0.5$$

$$\Rightarrow \exp\left(\frac{-3}{500} t\right) = 0.5$$

$$\Rightarrow \frac{-3}{500} t = \ln(0.5) \quad \therefore \boxed{t = \frac{500 \ln(0.5)}{3}}$$

Newton's Law of Cooling

The rate of change in temperature T at time t for some object in a media of temperature M is proportional to the difference in temperature from the object and media. In math,

$$\frac{dT}{dt} = K[M - T]$$

Example: Suppose a coffee had intial temperature 120 °F and after 10 minutes it cooled to 100°F find it's temperature at time t assuming it obeys Newton's Law of cooling in an ambient temperature of 70° F.

$$\frac{dT}{dt} = K[70 - T] = 70K - KT$$

$$\frac{dT}{dt} + KT = 70k \Rightarrow \underline{T(t) = Ce^{-Kt} + 70}$$

Now
$$T(0) = 120 = C + 70 \Rightarrow C = 50$$
$$T(10) = 100 = 50e^{-10K} + 70$$
$$\Rightarrow \frac{3}{5} = e^{-10K} \Rightarrow \ln(3/5) = -10K$$
$$\Rightarrow K = \frac{-1}{10}\ln(3/5)$$

$$\therefore \boxed{T(t) = 50e^{\frac{1}{10}\ln(3/5)t} + 70}$$

Remark: the text has a rather interesting analysis of temperature control in buildings. I've restricted the discussion to plain-old Newton's Law for time's sake, you might read § 3.3 to see a more sophisticated model of some everyday things.

NEWTONIAN MECHANICS

You may recall Newton's Laws of motion, paraphrasing a bit,

1.) A body subject to no external force has constant velocity
2.) A body subject to an external force has a time rate of change of momenta equal to the vector sum of the external force
3.) Equal and opposite reaction

Now I prefer,

1.) $\vec{F}_{ext} = 0 \Rightarrow \frac{d}{dt}(\vec{V}) = 0$

2.) $\vec{F}_{ext} = \frac{d\vec{P}}{dt}$

3.) $\vec{F}_{12} = -\vec{F}_{21}$ $\quad$ F_{12} is force of 1 on 2
F_{21} is force of 2 on 1

Where $\vec{F}_{ext}$ is the net external force on the object and $\vec{P} = m\vec{V}$ is the momentum of the object.
Usually 2.) looks like

$$F = \frac{d}{dt}(mV) = \frac{dm}{dt}V + m\frac{dV}{dt} \Rightarrow F = ma$$

$$\left(\text{when } \frac{dm}{dt} = 0\right)$$

To begin with we say $F = ma$ but in general $F = \frac{dP}{dt}$ says much more.

Remark: See pg. 174 $\boxed{E6}$ for an example of a problem with $\frac{dm}{dt} \neq 0$ (under the Calculus II notes)

$\boxed{E1}$ Suppose a ball is rolled on a track with friction proportional to the velocity. Suppose β is the proportionality constant and it's intial velocity is V_0,

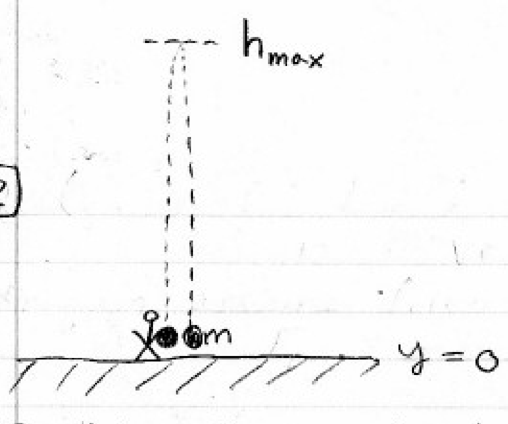
$$F = ma = -\beta V \Rightarrow m\frac{dV}{dt} = -\beta V$$

$$\Rightarrow \int \frac{dV}{V} = \int \frac{-\beta}{m}\, dt$$

$$\ln|V| = \frac{-\beta}{m}t + C_1 \Rightarrow \boxed{V = V_0 e^{\frac{-\beta}{m}t}}$$

the velocity slows exponentially to zero.

E2



imagine a ball is thrown vertically against the force of gravity. Assume friction $F_f = -bV$ for $b>0$ and $F_{gravity} = -mg$.

I take $y=0$ to be ground level and y increases as you go up. Newton's 2nd law yields,

$$m\frac{dV}{dt} = -mg - bV \quad , \quad V = \frac{dy}{dt}.$$

$$\frac{dV}{dt} + \frac{b}{m}V = -g \quad \Rightarrow \quad \mu = e^{\frac{b}{m}t} \quad \text{thus,}$$

$$\frac{d}{dt}\left(e^{\frac{bt}{m}}V\right) = -g e^{\frac{b}{m}t} \quad \Rightarrow \quad V(t) = C_1 e^{-\frac{bt}{m}} - gm/b$$

Call the intial velocity V_0 $\Rightarrow$ $\boxed{V(t) = \left(V_0 + gm/b\right)e^{-\frac{bt}{m}} - gm/b}$

When $b=0$ it's easier, $\frac{dV}{dt} = -g$ $\therefore$ $V = V_0 - gt$ and we can determine the time at which h_{max} is obtained by observing $V=0$ at h_{max} so $\boxed{t = V_0/g}$
Let's compare that to the time it takes for the ball with friction to reach h_{max}.

$$0 = \left(V_0 + gm/b\right)e^{-bt/m} - gm/b$$

$$\frac{g\cancel{m}}{\cancel{b}} = \left(V_0 + \frac{gm}{b}\right)e^{-bt/m}$$

$$= \left(V_0 + \frac{gm}{b}\right)\left(1 - \frac{bt}{m} + \cdots\right) \quad , \quad \text{keeping upto } 1^{st} \text{ order}$$

$$= V_0 - \frac{b}{m}tV_0 + \cancel{\frac{gm}{b}} - gt + \cdots$$

$$\Rightarrow V_0 = t\left(\frac{bV_0}{m} + g\right) \quad \therefore \quad \boxed{t \cong \frac{V_0}{\frac{bV_0}{m} + g}}$$

We see the time to summit of flight is shortened since $\frac{V_0}{g} > \frac{V_0}{\frac{bV_0}{m} + g}$ as $b, m, V_0 > 0$ by assumption.

Alternatively just solve, $\frac{gm/b}{V_0 + gm/b} = e^{-bt/m}$

$$\Rightarrow t = -\frac{m}{b}\ln\left(\frac{gm/b}{V_0 + gm/b}\right) = \frac{m}{b}\ln\left(\frac{V_0 b}{gm} + 1\right)$$

I didn't need power series here