Convolution

Def^n/ Let $f(t)$ and $g(t)$ be piecewise continous on $[0,\infty)$. The convolution of f and g is denoted $f*g$ and is defined by

$$(f*g)(t) \equiv \int_0^t f(t-v)\,g(v)\,dv$$

E1) $$t*t^2 = \int_0^t (t-v)v^2\,dv$$
$$= \int_0^t (tv^2 - v^3)\,dv$$
$$= \left(\tfrac{1}{3}tv^3 - \tfrac{1}{4}v^4\right|_0^t$$
$$= \tfrac{1}{3}t^4 - \tfrac{1}{4}t^4$$
$$= \boxed{\tfrac{1}{12}t^4 = t*t^2}$$

Th^m (10) Given piecewise continous fncts f, g, h on $[0,\infty)$ we have

1.) $f*g = g*f$
2.) $f*(g+h) = f*g + f*h$
3.) $f*(g*h) = f*(g*h)$
4.) $f*0 = 0$

Proof: see text.

Th^m (11) (Convolution Th^m) Given f, g piecewise continuous on $[0,\infty)$ and of exponential order α with Laplace transforms $\mathcal{L}\{f\}(s) = F(s)$ and $\mathcal{L}\{g\}(s) = G(s)$ then,

$$\mathcal{L}\{f*g\}(s) = F(s)\,G(s)$$

Or in other words

$$\mathcal{L}^{-1}\{FG\}(t) = (f*g)(t)$$

- Of course this Th^m is the whole reason for defining such a thing as a "convolution".

Proof: $\hookleftarrow$

Proof:

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty e^{-st}(f*g)(t)\,dt$$
$$= \int_0^\infty e^{-st}\left[\int_0^t f(t-v)\,g(v)\,dv\right]dt$$
$$= \int_0^\infty e^{-st}\left[\int_0^\infty u(t-v)f(t-v)\,g(v)\,dv\right]dt$$

added zero.

$$= \int_0^\infty g(v)\left[\int_0^\infty e^{-st}u(t-v)f(t-v)\,dt\right]dv$$

switched order of integration, technically needs some justification here.

$$= \int_0^\infty g(v)\,\mathcal{L}\{u(t-v)f(t-v)\}(s)\,dv$$
$$= \int_0^\infty g(v)\,e^{-sv}F(s)\,dv$$

used Th^m(8)

$$= F(s)\int_0^\infty e^{-sv}g(v)\,dv$$
$$= F(s)\,G(s).$$

$\boxed{E1}$ $y''+y = g(t)$ with $y(0)=0$ and $y'(0)=0$.

$s^2Y + Y = G$ taking the Laplace transform.

$(s^2+1)Y = G \Rightarrow Y(s) = \left(\frac{1}{s^2+1}\right)G(s)$

Use the convolution Th^m,

$$y(t) = \mathcal{L}^{-1}\{Y\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}G(s)\right\}(t)$$
$$= \left(\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} * \mathcal{L}^{-1}\{G\}\right)(t)$$
$$= \sin(t) * g(t) \quad \text{where } \mathcal{L}\{g\} = G.$$
$$= \boxed{\int_0^t \sin(t-v)\,g(v)\,dv = y(t)}$$

- This is an integral sol^n to the DEq^n. This result is quite impressive, notice it works for any piecewise continuous forcing term of exp. order.

[E2] $y'' + y = \cos t$, with $y(0) = y'(0) = 0$. We found that

$$y = \int_0^t \sin(t-v)\cos v \, dv$$

$$= \int_0^t (\sin t \cos^2 v - \sin v \cos t \cos v)dv$$

$$= \sin t \int_0^t \tfrac{1}{2}(1+\cos 2v)dv - \cos t \int_0^t \sin v \cos v dv$$

$$= \tfrac{1}{2}\sin t \left(v + \tfrac{1}{2}\sin 2v \Big|_0^t - \cos t \, \tfrac{1}{2}\sin^2 v \Big|_0^t\right.$$

$$= \tfrac{1}{2}\sin t\left(t + \tfrac{1}{2}\sin 2t\right) - \cos t\left(\tfrac{1}{2}\sin^2 t\right)$$

$$= \tfrac{1}{2}t\sin t + \cancel{\tfrac{1}{4}\sin t(2\sin t\cos t)} - \cancel{\tfrac{1}{2}\cos t \sin^2 t}$$

$$= \boxed{\tfrac{1}{2}t\sin t}$$ you can verify that $y(0) = y'(0) = 0$ as req^{d}.

<u>Remark</u>: this gives us yet another method to explain the presence of the factor "t" in y_p when there is overlap. Here $y_1 = \cos t$ & $y_2 = \sin t$ are the fundamental sol^{n}'s, clearly $\cos t = g(t)$ overlaps.

[E3] Solve $y'' - y = g(t)$ with $y(0) = 1$ and $y'(0) = 1$ assuming $g(t)$ has well defined $G(s) = \mathcal{L}\{g\}(s)$.

$$s^2 Y - sy(0) - y'(0) - Y = G$$

$$(s^2-1)Y = G + s + 1$$

$$\Rightarrow Y(s) = \frac{s+1}{s^2-1} + \frac{1}{s^2-1}G(s) = \frac{1}{s-1} + \frac{1}{s^2-1}G(s)$$

Notice by Partial Fractions $\frac{1}{s^2-1} = \frac{-1/2}{s+1} - \frac{1/2}{s-1}$ therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\}(t) = \tfrac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) - \tfrac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t)$$

$$= \tfrac{1}{2}(e^t - e^{-t}) \equiv \sinh(t)$$ ← hyperbolic sine function.

Hence, using the convolution Th^{m},

$$y(t) = e^t + \sinh(t) * g(t) = \boxed{e^t + \int_0^t \sinh(t-v)g(v)dv = y(t)}$$

E4 Use the convolution Th$^{\underline{m}}$ to find $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$.

$$\frac{1}{(s^2+1)^2} = \frac{1}{s^2+1}\,\frac{1}{s^2+1} = F(s)\,G(s)$$

Now $\mathcal{L}^{-1}\{F\}(t) = \sin t = f(t) = g(t) = \mathcal{L}^{-1}\{G\}(t)$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}(t) &= \mathcal{L}^{-1}\{FG\}(t) \\
&= f(t) * g(t) \\
&= \int_0^t f(t-v)\,g(v)\,dv \\
&= \int_0^t \sin(t-v)\sin(v)\,dv \\
&= \int_0^t (\sin t\cos v - \sin v\cos t)\sin v\,dv \\
&= \sin t\int_0^t \sin v\cos v\,dv - \cos t\int_0^t \sin^2 v\,dv \\
&= \sin t\left(\tfrac{1}{2}\sin^2 v\right)\Big|_0^t - \cos t\int_0^t \tfrac{1}{2}(1-\cos 2v)\,dv \\
&= \tfrac{1}{2}\sin^3 t - \cos t\left(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right) \\
&= \tfrac{1}{2}\sin^3 t - \tfrac{1}{2}t\cos t + \tfrac{1}{2}\cos^2 t\sin t \quad , \text{ used } \sin 2t = 2\sin t\cos t \\
&= \tfrac{1}{2}\sin t(\sin^2 t + \cos^2 t) - \tfrac{1}{2}t\cos t \\
&= \boxed{\tfrac{1}{2}\sin t - \tfrac{1}{2}t\cos t}
\end{aligned}$$

The text reached the same result in ex. 2 on p. 402, we used a different trig-identity.

Remark: the text discusses a concept of the <u>transfer function</u>. This is an interesting and probably useful concept for some of you. We will not however cover that concept in this course, it would be good to read over it if you have time.

Remark: the convolution Th$^{\underline{m}}$ allows us to unravel many <u>inverse</u> Laplace Transforms in a slick way. In the other direction it is not as useful since as a starting point you need to identify some convolution in t. Unless your example is very special (like ex. 3 on p. 402) it is unlikely the convol. Th$^{\underline{m}}$ will be usefull in taking $\mathcal{L}$.