MA 341 – HOMEWORK ONE SOLUTION – FALL 2007

PROBLEM ONE

§ 2.3 #20 Solve $\frac{dy}{dx} + \frac{3y}{x} + 2 = 3x$ given $y(1) = 1$.

To begin rewrite in standard form,

$$\frac{dy}{dx} + \left(\frac{3}{x}\right) y = 3x - 2$$

Identify that $P(x) = 3/x$. Calculate the integrating factor,

$$\begin{aligned} \mu &= \exp\left(\int \frac{3}{x}\,dx\right) \\ &= \exp(3 \ln|x|) \\ &= \exp(\ln|x|^3) \\ &= |x|^3 \end{aligned}$$

We can restrict our attention to $x > 0$ since that is where are initial condition resides, so $|x| = x$ and $\mu = x^3$. Multiply by μ as usual,

$$\underbrace{x^3 \frac{dy}{dx} + 3x^2 y}_{} = x^3(3x-2) = 3x^4 - 2x^3$$

$$\frac{d}{dx}(x^3 y) = 3x^4 - 2x^3 \quad : \text{used product rule.}$$

$$\int \frac{d}{dx}(x^3 y)\,dx = \int (3x^4 - 2x^3)\,dx \quad : \text{integrate}$$

$$x^3 y = \frac{3}{5}x^5 - \frac{2}{4}x^4 + C$$

$$y = \frac{3}{5}x^2 - \frac{1}{2}x + C \qquad (\text{general sol}^{\underline{n}})$$

We were given $y(1) = 1$ so we can find C,

$$1 = \frac{3}{5}(1)^2 - \frac{1}{2}(1) + C \quad \Rightarrow \quad C = 1 - \frac{3}{5} + \frac{1}{2} = \frac{9}{10}$$

$$\therefore \boxed{y = \frac{3}{5}x^2 - \frac{1}{2}x + \frac{9}{10}}$$

PROBLEM TWO

§2.4#24 Solve $(e^t x + 1)dt + (e^t - 1)dx = 0$ given $x(1) = 1$.

Notice $\partial M/\partial x = \partial/\partial x(e^t x + 1) = e^t$ and $\partial N/\partial t = \partial/\partial t(e^t - 1) = e^t$

thus $\partial M/\partial x = \partial N/\partial t$ and by the Th$^{\underline{m}}$ this is an exact eq$^{\underline{n}}$.

So we can find $F = F(x,t)$ such that,

$$\frac{\partial F}{\partial t} \overset{①}{=} e^t x + 1 \qquad \frac{\partial F}{\partial x} \overset{②}{=} e^t - 1$$

We can integrate ① w.r.t. t holding x fixed,

$$F = \int \frac{\partial F}{\partial t} dt = \int (e^t x + 1) dt = e^t x + t + C_1(x)$$

then substitute into ②,

$$\frac{\partial F}{\partial x} = e^t - 1 \Rightarrow e^t + \frac{\partial C_1}{\partial x} = e^t - 1$$

$$\Rightarrow \frac{\partial C_1}{\partial x} = -1 \Rightarrow \underline{C_1 = -x + C_2}$$

we find $\underline{F(x,t) = e^t x + t - x + C_2}$. Solutions have the form $e^t x + t - x = k$. We know $x(1) = 1$,

$$e^1(1) + 1 - 1 = k \quad \therefore \quad k = e$$

$$\boxed{e^t x + t - x = e}$$

PROBLEM 3

§3.3#2 $T(0) = 35$ and $T(3) = 40$. Room Temp $= 70$.

Newton's Law of Cooling says

$$\frac{dT}{dt} = k(T - 70)$$

we may solve via sep. of variables.

$$\frac{dT}{T-70} = k\,dt \Rightarrow \ln|T - 70| = kt + C_1$$

$$\Rightarrow |T - 70| = e^{kt + C_1} = C_2 e^{kt}$$

$$\Rightarrow T - 70 = \pm C_2 e^{kt}$$

$$\Rightarrow \underline{T(t) = 70 + C_3 e^{kt}}$$

PROBLEM 3 CONTINUED

§ 3.3#2 Continued We wish to find $T(20)$ but first we need to find $T(t)$. Use our data,

$$T(0) = 35 = 70 + C_3 \quad \therefore \quad C_3 = -35$$

$$T(3) = 40 = 70 - 35e^{3k}$$

$$\Rightarrow -30 = -35e^{3k}$$

$$\Rightarrow \frac{30}{35} = e^{3k}$$

$$\Rightarrow \frac{6}{7} = e^{3k}$$

$$\Rightarrow \ln(6/7) = 3k \quad \therefore \quad k = \tfrac{1}{3}\ln(6/7).$$

Thus $T(t) = 70 - 35e^{\frac{1}{3}\ln(6/7)t}$. So we can calculate,

$$T(20) = 70 - 35e^{\frac{1}{3}\ln(6/7)20}$$

$$= 70 - 35\exp\left(\ln\left([6/7]^{20/3}\right)\right)$$

$$= 70 - 35\cdot\left(\frac{6}{7}\right)^{20/3}$$

$$= \boxed{57.48}$$

Remark: We can calulate $k = -0.05138$. The quantity $\tau = |1/k| = 19.46$ is the "time constant" for this system. Systems with exponential decay are characterized by τ. For example a discharging capacitor will lose 64 % of its charge within the first τ seconds of discharge. This problem is similar, we were asked what is $T(20)$ well $t = 20 \approx \tau = 19.46$. We expect that 36 % of the difference in temperature from 70 has disapated, lets see if this checks with our answer,

$$70 - 57.48 = 12.52 \quad \text{and} \quad \frac{12.52}{35} \cong 0.36 \text{ aka } 36\%.$$

PROBLEM 4

§4.4 #18 (added instructions: Use annihilator method to "guess" y_p)

Find y_p for $y'' + 4y = 8\sin(2t)$. To begin note the aux. eq$^{\underline{n}}$ is

$$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm\sqrt{-4} = \pm 2i$$

$$\Rightarrow \underline{y_h = c_1\cos(2t) + c_2\sin(2t)}$$

Now we use the annihilator method to guess y_p. Notice that $g(t) = 8\sin(2t)$ we need $A[g] = 0$. Well we just found $\sin(2t)$ solves $y'' + 4y = 0$ so use

$$A = D^2 + 4$$

Don't believe it? Check,

$$\begin{aligned} A[8\sin(2t)] &= (D^2+4)[8\sin(2t)] \\ &= D^2[8\sin(2t)] + 32\sin(2t) \\ &= D[16\cos(2t)] + 32\sin(2t) \\ &= -32\sin(2t) + 32\sin(2t) \\ &= 0. \end{aligned}$$

So, we can convert the given nonhomogeneous DEq$^{\underline{n}}$ to a homog.

$$(D^2+4)[y] = 8\sin(2t) \Rightarrow A(D^2+4)[y] = A[8\sin(2t)] = 0$$

$$\Rightarrow (D^2+4)^2[y] = 0$$

$$\Rightarrow (\lambda^2+4)^2 = 0$$

$$\Rightarrow \lambda = \pm 2i \ \underline{\text{twice}}$$

$$\Rightarrow y = \underbrace{c_1\cos(2t) + c_2\sin(2t)}_{\text{we already know this is } y_h} + \underbrace{c_3 t\cos(2t) + c_4 t\sin(2t)}_{\text{this suggests we should use}}$$

$$\underline{y_p = At\cos(2t) + Bt\sin(2t)}$$

we're not done yet

(continued)

PROBLEM 4

§4.4 #18 We found $y_p = At\cos(2t) + Bt\sin(2t)$

factoring: $y_p = t(A\cos(2t) + B\sin(2t))$.

$$y_p' = A\cos 2t + B\sin 2t + t(-2A\sin 2t + 2B\cos 2t)$$
$$y_p' = \cos(2t)[A + 2Bt] + \sin(2t)[B - 2At]$$

$$y_p'' = -2\sin(2t)[A+2Bt] + \cos(2t)[2B] + 2\cos(2t)[B-2At] + \sin(2t)[-2A]$$
$$= \cos(2t)[2B + 2(B - 2At)] + \sin(2t)[-2[A+2Bt] - 2A]$$
$$= \cos(2t)[4B - 4At] + \sin(2t)[-4A - 4Bt]$$

Now substitute to determine the undetermined coefficients,

$$y_p'' + 4y_p = 8\sin(2t)$$

$$\cos(2t)[4B - 4At] + \sin(2t)[-4A - 4Bt] + 4At\cos(2t) + 4Bt\sin(2t) = 8\sin(2t)$$
$$\cos(2t)[4B \cancel{- 4At} \cancel{+ 4At}] + \sin(2t)[-4A \cancel{- 4Bt} \cancel{+ 4Bt}] = 8\sin(2t)$$

Equate coefficients of $\cos(2t)$ & $\sin(2t)$ to find,

$$4B = 0 \Rightarrow B = 0$$
$$-4A = 8 \Rightarrow A = -2$$

Thus

$$\boxed{y_p = -2t\cos(2t)}$$

PROBLEM 5

§4.5 #20 (I added instruction to use annihilator method)

Solve $y'' + 4y = \sin\theta - \cos\theta$. Here θ is the independent variable so $y'' = \frac{d^2y}{d\theta^2}$ etc.... Find y_h,

$$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i \Rightarrow y_h = c_1\cos(2\theta) + c_2\sin(2\theta)$$

Next annihilate $g(\theta) = \sin\theta - \cos\theta$. Well, I know $\sin\theta$ & $\cos\theta$ solve $y'' + y = 0$ so use $A = D^2 + 1$. Let's check that it works,

$$\begin{aligned} A[\sin\theta - \cos\theta] &= (D^2+1)[\sin\theta - \cos\theta] \\ &= D^2[\sin\theta - \cos\theta] + 1[\sin\theta - \cos\theta] \\ &= -\cancel{\sin\theta} + \cancel{\cos\theta} + \cancel{\sin\theta} - \cancel{\cos\theta} \\ &= 0. \end{aligned}$$

Next convert $y'' + 4y = \sin\theta - \cos\theta$ to $(D^2+4)[y] = \sin\theta - \cos\theta$ and annihilate it,

$$\underbrace{(D^2+1)(D^2+4)[y]}_{} = (D^2+1)[\sin\theta - \cos\theta] = 0.$$

Sol$^{\underline{n}}$'s are $y = \underbrace{c_1\cos\theta + c_2\sin\theta}_{\text{suggests}} + \underbrace{c_3\cos(2\theta) + c_4\sin(2\theta)}_{\text{this is } y_h}$

$$y_p = A\cos\theta + B\sin\theta.$$

Then calculate,

$$\begin{aligned} y_p' &= -A\sin\theta + B\cos\theta \\ y_p'' &= -A\cos\theta - B\sin\theta \end{aligned}$$

And Substitute into $y_p'' + 4y_p = \sin\theta - \cos\theta$,

$$-A\cos\theta - B\sin\theta + 4A\cos\theta + 4B\sin\theta = \sin\theta - \cos\theta$$

Equate Coefficients $\Rightarrow$ $3A = -1$ and $3B = 1$ thus,

$$\boxed{y = y_h + y_p = c_1\cos(2\theta) + c_2\sin(2\theta) - \frac{1}{3}\cos(\theta) + \frac{1}{3}\sin(\theta)}$$

PROBLEM 6

§4.6 #10 Solve $y'' + 4y' + 4y = e^{-2t}\ln(t)$.

$$\lambda^2 + 4\lambda + 4 = 0 \Rightarrow (\lambda+2)^2 = 0 \therefore \lambda_1 = -2 = \lambda_2$$

The fundamental solⁿ's are $y_1 = e^{-2t}$ and $y_2 = te^{-2t}$.
We employ variation of parameters to look for solⁿ of form $y_p = e^{-2t}V_1 + te^{-2t}V_2$. Find V_1 & V_2 using Eqⁿ (10) on pg. 195 of text, here $a=1$, $g = e^{-2t}\ln(t)$
Calculate the denominator to start,

$$\begin{aligned} y_1y_2' - y_2y_1' &= e^{-2t}(e^{-2t} - 2te^{-2t}) - te^{-2t}(-2e^{-2t}) \\ &= e^{-4t} - \cancel{2te^{-4t}} \cancel{+2te^{-4t}} \\ &= e^{-4t} \end{aligned}$$

Thus, using Eqⁿ (10),

$$V_1 = \int \frac{-gy_2}{y_1y_2' - y_2y_1'}\,dt = \int \frac{-e^{-2t}\ln(t)\,te^{-2t}}{e^{-4t}}\,dt = -\int \underbrace{\ln(t)}_{u}\,\underbrace{t\,dt}_{dV}$$

$$= -\left(uV - \int V\,du\right)$$ where: $u = \ln(t)$, $dV = t\,dt$: Integration by Parts; $du = dt/t$, $V = \frac{1}{2}t^2$

$$= -\left(\frac{1}{2}t^2\ln(t) - \int \frac{1}{2}t^2\frac{dt}{t}\right)$$

$$= -\frac{1}{2}t^2\ln(t) + \frac{1}{4}t^2.$$, no need to add constants here we already have it in y_h later.

$$\begin{aligned} V_2 &= \int \frac{gy_1}{y_1y_2' - y_2y_1'}\,dt \\ &= \int e^{4t}e^{-2t}\ln(t)e^{-2t}\,dt \\ &= \int \ln(t)\,dt \end{aligned}$$: let $u = \ln(t)$, $dV = dt \Rightarrow du = \frac{dt}{t}$ & $V = t$

$$\begin{aligned} &= t\ln(t) - \int t\frac{dt}{t} \\ &= t\ln(t) - t. \end{aligned}$$

Thus we find,

$$\boxed{\begin{aligned} y &= c_1e^{-2t} + c_2te^{-2t} - \left(\frac{1}{2}t^2\ln(t) - \frac{1}{4}t^2\right)e^{-2t} + (t\ln t - t)te^{-2t} \\ &= c_1e^{-2t} + c_2te^{-2t} + \left(\frac{1}{2}t^2\ln(t) - \frac{3}{4}t^2\right)e^{-2t} \end{aligned}}$$

PROBLEM 7

§6.2 #14

$$y'''' + 2y''' + 10y'' + 18y' + 9y = 0$$
$$\lambda^4 + 2\lambda^3 + 10\lambda^2 + 18\lambda + 9 = 0$$

we are given that $y = \sin(3x)$ is a sol$^{\underline{n}}$. This reveals that our DEq$^{\underline{n}}$ is $(D^2+9)(\text{stuff})[y] = 0$ now because it's $4^{\underline{th}}$ order and the coefficient of y'''' is one we can factor the auxx. eq$^{\underline{n}}$ need to find

$$\lambda^4 + 2\lambda^3 + 10\lambda^2 + 18\lambda + 9 = (\lambda^2+9)(\lambda^2 + B\lambda + C)$$
$$= \lambda^4 + B\lambda^3 + C\lambda^2 + 9\lambda^2 + 9B\lambda + 9C$$
$$= \lambda^4 + \lambda^3(B) + \lambda^2(C+9) + \lambda(9B) + 9C$$

Equating coefficients, $B = 2$ and $C+9 = 10$, $9B = 18$, $9 = 9C$
So clearly $B = 2$ and $C = 1$. Therefore,

$$0 = \lambda^4 + 2\lambda^3 + 10\lambda^2 + 18\lambda + 9 = (\lambda^2+9)(\lambda^2 + 2\lambda + 1)$$
$$= (\lambda^2+9)(\lambda+1)^2 = 0$$

thus $\lambda_{1,2} = \pm 3i$ and $\lambda_3 = -1 = \lambda_4$. We find

$$\boxed{y_h = c_1 \cos(3x) + c_2 \sin(3x) + c_3 e^{-x} + c_4 x e^{-x}}$$

PROBLEM 8

§6.2 #18

$(D-1)^3(D-2)(D^2+D+1)(D^2+6D+10)^3[y] = 0$. The aux eq$^{\underline{n}}$ is $(\lambda-1)^3(\lambda-2)(\lambda^2+\lambda+1)(\lambda^2+6\lambda+10)^3 = 0$.

Notice $\lambda^2 + \lambda + 1 = 0 \Rightarrow \lambda = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$

$\lambda^2 + 6\lambda + 10 = 0 \Rightarrow \lambda = \frac{-6 \pm \sqrt{36-40}}{2} = -3 \pm i$

Thus we find,

$$\boxed{\begin{aligned} y_h = {} & c_1 e^x + c_2 x e^x + c_3 x^2 e^x + c_4 e^{2x} \\ & + c_5 e^{-\frac{1}{2}x} \cos\left(\tfrac{\sqrt{3}}{2}x\right) + c_6 e^{-\frac{1}{2}x} \sin\left(\tfrac{\sqrt{3}}{2}x\right) \\ & + c_7 e^{-3x}\cos(x) + c_8 e^{-3x}\sin(x) + c_9 x e^{-3x}\cos(x) + c_{10} x e^{-3x}\sin(x) \\ & + c_{11} x^2 e^{-3x}\cos(x) + c_{12} x^2 e^{-3x}\sin(x) \end{aligned}}$$

PROBLEM 9

§6.3 # 16 Annihilate $x^2 - e^x$. That is find a smooth operator A such that $A[x^2 - e^x] = 0$. Well, clearly $D^3 x^2 = 0$ and $(D-1)e^x = 0$ hence

$$\boxed{A = D^3(D-1)}$$

let's check it,

$$\begin{aligned} A[x^2 - e^x] &= D^3(D-1)[x^2 - e^x] \\ &= D^3(D-1)[x^2] - D^3(D-1)[e^x] \\ &= (D-1)D^3[x^2] - D^3[0] \\ &= (D-1)[0] \\ &= 0. \end{aligned}$$

you can explicitly use $D = d/dx$ if my methods here disturb you. The important thing is that you can correctly find A.

PROBLEM 10 Simply use $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and algebra,

$$\begin{aligned} \cos(x)\cos(2x) &= \tfrac{1}{2}(e^{ix} + e^{-ix})\tfrac{1}{2}(e^{2ix} + e^{-2ix}) \\ &= \tfrac{1}{2}\tfrac{1}{2}(e^{3ix} + e^{-ix} + e^{ix} + e^{-3ix}) \\ &= \tfrac{1}{2}\left[\tfrac{1}{2}(e^{3ix} + e^{-3ix}) + \tfrac{1}{2}(e^{ix} + e^{-ix})\right] \\ &= \tfrac{1}{2}\left[\cos(3x) + \cos(x)\right]. \end{aligned}$$