# SOLUTION TO HOMEWORK ON SYSTEMS OF ODES

$\boxed{\S 9.1\#12}$

$\boxed{\text{PROBLEM } 1}$ Convert $\begin{array}{l} x'' + 3x' - y' + 2y = 0 \\ y'' + x' + 3y' + y = 0 \end{array}$ to a system in normal form. Let us define $x_1 = x$, $x_2 = x'$, $x_3 = y$, $x_4 = y'$

$$x'' = x_2' = -3x' + y' - 2y = -3x_2 + x_4 - 2x_3 = x_2'$$
$$y'' = x_4' = -x' - 3y' - y = -x_2 - 3x_4 - x_3 = x_4'$$

of course we also have $x_1' = x_2$ and $x_3' = x_4$. In matrix form,

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}$$

- depending on your choice for $x_1, x_2, x_3, x_4$ your answer could look different.

Now $\boxed{\S 9.2\#9}$. Solve the following,

$$\begin{array}{rl} (1-i)x_1 + 2x_2 = 0 & \text{Eq}^{n}(I) \\ \underbrace{-x_1 - (1+i)x_2 = 0}_{x_1 = -(1+i)x_2} & \text{Eq}^{n}(II) \end{array} \longrightarrow \underbrace{\begin{bmatrix} 1-i & 2 \\ -1 & -(1+i) \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

I'll use substitution, $x_1 = -(1+i)x_2$ then subst. into $\text{Eq}^{n}(I)$.

$$(1-i)\left(-(1+i)x_2\right) + 2x_2 = 0$$
$$-(1 + i - i - i^2)x_2 + 2x_2 = 0$$
$$-2x_2 + 2x_2 = 0 \qquad \text{hmm...}$$

the sol$^{n}$ is thus $\boxed{x_2 = t \text{ and } x_1 = -(1+i)t \text{ for any } t.}$ there are infinitely many sol$^{n}$'s. Notice

$$\det(A) = \det\begin{pmatrix} 1-i & 2 \\ -1 & -(1+i) \end{pmatrix} = -(1-i)(1+i) + 2 = -2 + 2 = 0.$$

We could have anticipated the lack of a unique sol$^{n}$.

$\boxed{\text{PROBLEM 2}}$ Find the inverse of $X(t)=\begin{bmatrix}\sin(2t) & -\cos(2t)\\ 2\cos(2t) & 2\sin(2t)\end{bmatrix}$

this is a $2\times2$ we can use the formula I proved in class,

$$X^{-1}(t)=\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

$$=\frac{1}{2\sin^2(2t)+2\cos^2(2t)}\begin{bmatrix}2\sin(2t) & \cos(2t)\\ -2\cos(2t) & \sin(2t)\end{bmatrix}$$

$$=\boxed{\frac{1}{2}\begin{bmatrix}2\sin 2t & \cos 2t\\ -2\cos 2t & \sin 2t\end{bmatrix}=X^{-1}(t)}$$

Next to find $Y^{-1}(t)$ for $Y(t)=\begin{bmatrix}e^{3t} & 1 & t\\ 3e^{3t} & 0 & 1\\ 9e^{3t} & 0 & 0\end{bmatrix}$. I'll use the TI-89,

$$\boxed{Y^{-1}(t)=\begin{bmatrix}0 & 0 & \frac{1}{9}e^{-3t}\\ 1 & -t & \frac{t}{3}-\frac{1}{9}\\ 0 & 1 & -1/3\end{bmatrix}}$$

Maple, Matlab Mathematica etc... also can do this sort of calc.

If technology make you angry you could also use the standard trick,

$$\left[\begin{array}{ccc|ccc}e^{3t} & 1 & t & 1 & 0 & 0\\ 3e^{3t} & 0 & 1 & 0 & 1 & 0\\ 9e^{3t} & 0 & 0 & 0 & 0 & 1\end{array}\right]\sim\left[\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\right|\left.\underbrace{\begin{matrix}0 & 0 & \frac{1}{9}e^{-3t}\\ 1 & -t & \frac{1}{3}t-\frac{1}{9}\\ 0 & 1 & \frac{-1}{3}\end{matrix}}_{Y^{-1}}\right]$$

where by $\sim$ I mean a series of row operations, like pg. (80) $\boxed{E8}$ in my notes.

(this problem is § 9.3 #18 & #20)

$\boxed{\text{PROBLEM 3}}$ (§9.4 #26) Let $\Phi(t)$ be a fundamental matrix for the system $\vec{x}' = A\vec{x}$. <u>Show</u> that $\vec{x}(t) = \Phi(t)\,\Phi^{-1}(t_0)\vec{x}_0$ is <u>the</u> sol$^{\underline{n}}$ to the initial value problem $\vec{x}' = A\vec{x}$, $\vec{x}(t_0) = \vec{x}_0$.

To begin since $\Phi(t)$ is a fundamental matrix we have <u>two</u> things,

$$\underbrace{\Phi' = A\Phi}_{\underset{\text{each is sol}^{\underline{n}}}{\Phi = [x_1|x_2|\cdots|x_n]}} \quad \text{and} \quad \underbrace{\det(\Phi) \neq 0}_{\underset{\text{are linearly independent sol}^{\underline{n}}\text{'s.}}{x_1, x_2, \dots, x_n}}$$

Now $\det(\Phi) \neq 0 \Rightarrow \Phi^{-1}$ exists. Let's check the claim,

$$\frac{d\vec{x}}{dt} = \frac{d}{dt}\left[\Phi(t)\,\Phi^{-1}(t_0)\vec{x}_0\right] \quad : \text{ use } \underline{\text{matrix}} \text{ product rule.}$$

$$= \frac{d\Phi}{dt}\Phi^{-1}(t_0)\vec{x}_0 + \cancel{\Phi(t)\frac{d}{dt}\Big(\overbrace{\Phi^{-1}(t_0)\vec{x}_0}^{\text{constant}}\Big)} \rightarrow 0$$

$$= A\,\Phi^{-1}(t_0)\vec{x}_0$$

$$= A\vec{x} \qquad \text{thus it is a sol}^{\underline{n}}.$$

We also need to check if $\vec{x}(t_0) = \vec{x}_0$,

$$\vec{x}(t_0) = \Phi(t_0)\,\Phi^{-1}(t_0)\vec{x}_0 = I\vec{x}_0 = \vec{x}_0. \;//$$

(§9.4 #28) $\vec{x}' = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\vec{x}$ and $\vec{x}(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Check if $\Phi(t) = \begin{bmatrix} e^{-t} & e^{5t} \\ -e^{-t} & e^{5t} \end{bmatrix}$ is a fundamental matrix, if so use #26 to write sol$^{\underline{n}}$ down.

$$\Phi' = \begin{bmatrix} -e^{-t} & 5e^{5t} \\ e^{-t} & 5e^{5t} \end{bmatrix} \quad \text{just differentiate component-wise.}$$

$$A\Phi = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} e^{-t} & e^{5t} \\ -e^{-t} & e^{5t} \end{bmatrix} = \begin{bmatrix} 2e^{-t} - 3e^{-t} & 2e^{5t} + 3e^{5t} \\ 3e^{-t} - 2e^{-t} & 3e^{5t} + 2e^{5t} \end{bmatrix} = \Phi'$$

So it is a sol$^{\underline{n}}$ notice $\det(\Phi) = e^{-t}e^{5t} + e^{5t}e^{-t} = 2e^{4t}$. then we calculate $\Phi^{-1}(t) = \frac{1}{2e^{4t}}\begin{bmatrix} e^{5t} & -e^{5t} \\ e^{-t} & e^{-t} \end{bmatrix}$ $\therefore$ $\Phi^{-1}(0) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

$$\vec{x}(t) = \begin{bmatrix} e^{-t} & e^{5t} \\ -e^{-t} & e^{5t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{-t} & e^{5t} \\ -e^{-t} & e^{5t} \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(4e^{-t} + 2e^{5t}) \\ \frac{1}{2}(-4e^{-t} + 2e^{5t}) \end{bmatrix}$$

$$\therefore \boxed{\vec{x}(t) = [\,2e^{-t} + e^{5t},\ -2e^{-t} + e^{5t}\,]^T}$$

**PROBLEM 4** Find eigenvalues & eigenvectors of $A = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}$.

$$\det(A - \lambda I) = \det\begin{pmatrix} 6-\lambda & -3 \\ 2 & 1-\lambda \end{pmatrix} = (6-\lambda)(1-\lambda) + 6$$

$$= \lambda^2 - 7\lambda + 12$$

$$= (\lambda - 3)(\lambda - 4) = 0 \quad \therefore \underline{\lambda_1 = 3 \ \& \ \lambda_2 = 4}$$

Find $\vec{u}_1 = (u, v)^T$ such that $(A - 3I)\vec{u}_1 = 0$, that is solve,

$$\begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} 3u - 3v = 0 \\ u = v \\ \text{choose } u = 1 \end{matrix} \rightarrow \boxed{\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Find $\vec{u}_2 = (u, v)^T$ such that $(A - 4I)\vec{u}_2 = 0$, that is solve,

$$\begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} 2u - 3v = 0 \\ u = \frac{3}{2}v \\ \text{choose } v = 2 \end{matrix} \rightarrow \boxed{\vec{u}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}}$$

Now we may solve $\vec{x}' = A\vec{x}$ according to the general results derived in lecture,

$$\boxed{\vec{x} = c_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 3 \\ 2 \end{bmatrix}}$$

(this is §9.5 #2)

$\boxed{\text{PROBLEM 5}}$ Find eigenvalues & eigenvectors of $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)\det\begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} - 2(0) - 1(0)$$

$$= (1-\lambda)\left[(1-\lambda)(1-\lambda) + 1\right]$$

$$= (1-\lambda)\left[\lambda^2 - 2\lambda + 2\right]$$

$$\underline{\lambda_1 = 1}, \quad \lambda_2 = \frac{2 \pm \sqrt{4-8}}{2} = \underline{1 \pm i = \lambda_2}$$

lets find the real eigenvector $\vec{u}_1$ with $(A-I)\vec{u}_1 = 0$

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} 2v - w &= 0 \\ w &= 0 \\ -v &= 0 \end{aligned} \qquad \boxed{\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

choose $u = 1$

next find the complex eigenvector $\vec{u}_2$ with $(A - (1+i)I)\vec{u}_2 = 0$.

$$\begin{bmatrix} 1-(1+i) & 2 & -1 \\ 0 & 1-(1+i) & 1 \\ 0 & -1 & 1-(1+i) \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & 2 & -1 \\ 0 & -i & 1 \\ 0 & -1 & -i \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-iu + 2v - w = 0 \longrightarrow iu = 2v - w = 2v - iv$$

$$-iv + w = 0$$

$$-v - iw = 0 \longrightarrow iv + i^2 w = 0 \rightarrow \underline{w = iv}$$

We find $u = \frac{1}{i}(2-i)v = (-2i-1)v$ and $w = iv$. Choose $v=1$,

PROBLEM 5 continued

We let $v=1$ for convenience and find

$$\vec{u}_2 = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -2i-1 \\ 1 \\ i \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{a}_2 = Re(\vec{u}_2)} + i \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{b}_2 = Im(\vec{u}_2)}$$

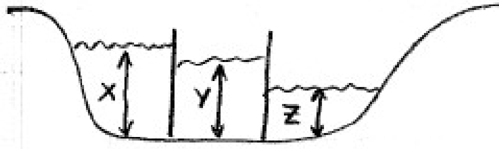
Thus we find the general sol$^{n}$,

$$\boxed{\begin{aligned}\vec{x} = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &+ c_2\left(e^{t}\cos(t) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - e^{t}\sin t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right) \\ &+ c_3\left(e^{t}\sin t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + e^{t}\cos t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right)\end{aligned}}$$

The reason for the funny form of the last two was derived in lecture, in short we want real sol$^{n}$'s but the math is easier to do with complex numbers. So when we find the complex sol$^{n}$ we have to extract two real sol$^{n}$'s.

(this is §9.5 #10)

**PROBLEM 6** Complete §9.5 #50, let $x, y, z$ be the water levels in the cells of the icetray pictured below. The water level will change at a rate proportional to the difference between the cell's water level and the adjacent cell's level. (I suppose the walls have holes so the water can level out, this is not like my ice tray at home, anyway continue)



assuming proportionality constant is one gives

$$\frac{dx}{dt} = Y - X \qquad \frac{dy}{dt} = (X-Y)+(Z-Y) \qquad \frac{dz}{dt} = Y - Z$$

I suppose these are reasonable, $\frac{dx}{dt} < 0$ when $x > y$ so water flows to lower level and vice-versa $\frac{dx}{dt} > 0$ when $x < y$. The remaining question is a physical one and I'd need more info to go further.

(b.) $$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \underbrace{\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longleftrightarrow \vec{x}' = A\vec{x}$$

$$0 = \det(A - \lambda I) = \det\begin{pmatrix} -1-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{pmatrix}$$

$$= (-1-\lambda)\det\begin{pmatrix} -2-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} - 1\det\begin{pmatrix} 1 & 1 \\ 0 & -1-\lambda \end{pmatrix} + 0$$

$$= -(\lambda+1)\left[(\lambda+2)(\lambda+1) - 1\right] + (\lambda+1)$$

$$= -(\lambda+1)\left[\lambda^2 + 3\lambda + 2 - 1 - 1\right]$$

$$= -(\lambda+1)\lambda(\lambda+3) = 0 \Rightarrow \underline{\lambda_1 = -1,\ \lambda_2 = 0,\ \lambda_3 = -3}$$

PROBLEM 6 Continued

Found eigenvalues $\lambda_1 = -1$, $\lambda_2 = 0$, $\lambda_3 = -3$ now find their respective eigenvectors $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$,

$$(A+I)\vec{u}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} v = 0 \\ u - v + w = 0 \Rightarrow w = -u \end{matrix}$$

choose $u = 1$.

$$\boxed{\vec{u}_1 = [1, 0, -1]^T}$$

$$A\vec{u}_2 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -u + v = 0 \rightarrow v = u \\ v - w = 0 \rightarrow w = v = u \end{cases}$$

choose $u = 1 = v = w$

$$\boxed{\vec{u}_2 = [1, 1, 1]^T}$$

$$(A+3I)\vec{u}_3 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2u + v = 0 \rightarrow \underline{v = -2u} \\ u + v + w = 0 \\ v + 2w = 0 \rightarrow v = -2w = -2u \end{cases}$$

choose $u = 1$, $\quad w = u$

$$\boxed{\vec{u}_3 = [1, -2, 1]^T}$$

The general sol$^{\underline{n}}$ is,

$$\vec{x}(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{-3t}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + c_2 + c_3 e^{-3t} \\ c_2 - 2c_3 e^{-3t} \\ -c_1 e^{-t} + c_2 + c_3 e^{-3t} \end{bmatrix}$$

Now we can fit $c_1, c_2, c_3$ to the initial conditions,

$$\left.\begin{matrix} x(0) = 3 = c_1 + c_2 + c_3 \\ y(0) = 0 = c_2 - 2c_3 \\ z(0) = 0 = -c_1 + c_2 + c_3 \end{matrix}\right\rangle \rightarrow 3 = 2c_1 \Rightarrow \boxed{c_1 = 3/2}$$

$$3/2 = c_2 + c_3 \therefore c_2 = 3/2 - c_3$$

$$0 = 3/2 - c_3 - 2c_3 = 3/2 - 3c_3 \Rightarrow \boxed{c_3 = \tfrac{1}{2}}$$

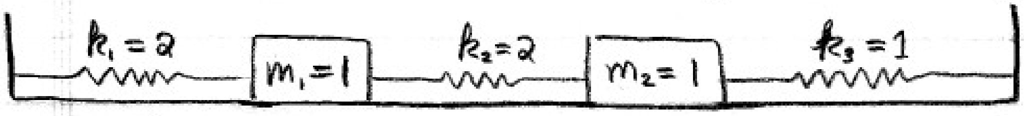
$$\& \boxed{c_2 = 1}$$

Thus,

$$\boxed{\begin{aligned} x(t) &= (3/2)e^{-t} + 1 + (1/2)e^{-3t} \\ y(t) &= 1 - e^{-3t} \\ z(t) &= (-3/2)e^{-t} + 1 + (1/2)e^{-3t} \end{aligned}}$$

Naturally $x(t), y(t), z(t)$ all go to 1 as $t \rightarrow \infty$, it's nice that they level out.

**PROBLEM 7** (§9.6 #19) Consider the following coupled spring system



Newton's 2nd law on $m_1$ and $m_2$ yield the following,

$$\left.\begin{aligned} x_1'' &= -2x_1 + 2(x_2 - x_1) \\ x_2'' &= -2(x_2 - x_1) - 3x_2 \end{aligned}\right\} (*)$$

I'll let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$, $y_4 = x_2'$ because I can. Then $(*)$ becomes,

$$\begin{aligned} y_1' &= y_3 \\ y_2' &= y_4 \\ y_3' &= -2y_1 + 2(y_2 - y_1) = -4y_1 + 2y_2 \\ y_4' &= -2(y_2 - y_1) - 3y_2 = 2y_1 - 5y_2 \end{aligned}$$

In matrix form we have

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 2 & 0 & 0 \\ 2 & -5 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \leftrightarrow \vec{y}\,' = A\vec{y}$$

Let's find the eigenvalues for A, these are the normal frequencies.

$$0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -4 & 2 & -\lambda & 0 \\ 2 & -5 & 0 & -\lambda \end{vmatrix}$$

$$= -\lambda \cdot \begin{vmatrix} -\lambda & 0 & 1 \\ 2 & -\lambda & 0 \\ -5 & 0 & -\lambda \end{vmatrix} - 0 + 1 \cdot \begin{vmatrix} 0 & -\lambda & 1 \\ -4 & 2 & 0 \\ 2 & -5 & -\lambda \end{vmatrix} - 0$$

$$\begin{aligned} &= -\lambda\left(-\lambda(\lambda^2) + 1(-5\lambda)\right) + \lambda(4\lambda) + 1(20 - 4) \\ &= -\lambda(-\lambda^3 - 5\lambda) + 4\lambda^2 + 16 \\ &= \lambda^4 + 5\lambda^2 + 4\lambda^2 + 16 \\ &= \lambda^4 + 9\lambda^2 + 16 \end{aligned}$$

$$\longrightarrow \lambda^2 = \frac{-9 \pm \sqrt{81 - 64}}{2} = \frac{-9 \pm \sqrt{17}}{2}$$

(quadratic formula for $\lambda^2$)

$$|\lambda| = \sqrt{\frac{-9 \pm \sqrt{17}}{2}} = \underbrace{i\sqrt{\frac{9 \mp \sqrt{17}}{2}}}_{\text{angular freq's.}} \Rightarrow \boxed{\begin{aligned} f_1 &= \frac{1}{2\pi}\sqrt{\frac{9 - \sqrt{17}}{2}} \cong 0.249 \\ f_2 &= \frac{1}{2\pi}\sqrt{\frac{9 + \sqrt{17}}{2}} \cong 0.408 \end{aligned}}$$

Remark: aren't you glad I asked for §9.6#19 instead of #20.

$\boxed{\text{PROBLEM 8}}$ Consider §5.2 # 38 the ARMS RACE

Let x and y be the expenditure of $'s on the military then ≈

$$\frac{dx}{dt} = -x + 2y + a$$

$$\frac{dy}{dt} = 4x - 3y + b$$

where a and b are constants that measure the trust (or distrust) of the nations with each other. What happens as $t \to \infty$ ?

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$$\vec{x}\,' = \begin{bmatrix} -1 & 2 \\ 4 & -3 \end{bmatrix}\vec{x} + \begin{bmatrix} a \\ b \end{bmatrix} \leftrightarrow \vec{x}\,' = A\vec{x} + \vec{f}$$

Find the fundamental matrix by solving $\vec{x}\,' = A\vec{x}$ to begin then we'll use variation of parameters to find sol$^{\underline{n}}$ overall.

$$0 = \det(A - \lambda I) = \det\begin{pmatrix} -1-\lambda & 2 \\ 4 & -3-\lambda \end{pmatrix} = (-1-\lambda)(-3-\lambda) - 8$$
$$= 3 + \lambda + 3\lambda + \lambda^2 - 8$$
$$= \lambda^2 + 4\lambda - 5$$
$$= (\lambda - 1)(\lambda + 5) \rightarrow \underline{\lambda_1 = 1, \lambda_2 = -5}$$

Find the eigenvectors $\vec{u}_1$ and $\vec{u}_2$ to match $\lambda_1 = 1$ and $\lambda_2 = -5$,

$$(A - I)\vec{u}_1 = \begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} -2u + 2v = 0 \\ u = v \\ \text{choose } u = 1 \end{matrix} \rightarrow \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\xrightarrow{\text{we see}} \vec{x}_1 = e^t\vec{u}_1 = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

$$(A + 5I)\vec{u}_2 = \begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} 4u + 2v = 0 \\ v = -2u \\ \text{choose } u = 1. \end{matrix} \rightarrow \vec{u}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\xrightarrow{\text{we see}} \vec{x}_2 = e^{-5t}\vec{u}_2 = \begin{bmatrix} e^{-5t} \\ -2e^{-5t} \end{bmatrix}$$

We find $$\Phi(t) = [\vec{x}_1 | \vec{x}_2] = \underbrace{\begin{bmatrix} e^t & e^{-5t} \\ e^t & -2e^{-5t} \end{bmatrix}}_{\text{half the battle's done.}} = \Phi(t)$$

**PROBLEM 8** Variation of Parameters for $\vec{x}' = A\vec{x} + \vec{f}$ said that $\vec{x} = \mathbb{X}\vec{c} + \mathbb{X}\int \mathbb{X}^{-1}\vec{f}\,dt$. We calculate with that formula in mind,

$$\mathbb{X}^{-1}(t) = \frac{1}{-2e^{t}e^{-5t} - e^{-5t}e^{t}}\begin{bmatrix} -2e^{-5t} & -e^{-5t} \\ -e^{t} & e^{t} \end{bmatrix} = \frac{e^{4t}}{3}\begin{bmatrix} 2e^{-5t} & e^{-5t} \\ e^{t} & -e^{t} \end{bmatrix}$$

$$\mathbb{X}^{-1}(t) = \frac{1}{3}\begin{bmatrix} 2e^{-t} & e^{-t} \\ e^{5t} & -e^{5t} \end{bmatrix}$$

$$\mathbb{X}^{-1}\vec{f} = \frac{1}{3}\begin{bmatrix} 2e^{-t} & e^{-t} \\ e^{5t} & -e^{5t} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2ae^{-t} + be^{-t} \\ ae^{5t} - be^{5t} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} (2a+b)e^{-t} \\ (a-b)e^{5t} \end{bmatrix}$$

$$\int \mathbb{X}^{-1}\vec{f}\,dt = \int \frac{1}{3}\begin{bmatrix} (2a+b)e^{-t} \\ (a-b)e^{5t} \end{bmatrix}dt = \frac{1}{3}\begin{bmatrix} (2a+b)\int e^{-t}dt \\ (a-b)\int e^{5t}dt \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -(2a+b)e^{-t} \\ \frac{1}{5}(a-b)e^{5t} \end{bmatrix}$$

$$\vec{x}_p = \mathbb{X}\int \mathbb{X}^{-1}\vec{f}\,dt = \frac{1}{3}\begin{bmatrix} e^{t} & e^{-5t} \\ e^{t} & -2e^{-5t} \end{bmatrix}\begin{bmatrix} -(2a+b)e^{-t} \\ \frac{1}{5}(a-b)e^{5t} \end{bmatrix}$$

notice the exponentials cancel out nicely.

$$= \frac{1}{3}\begin{bmatrix} -(2a+b) + \frac{1}{5}(a-b) \\ -(2a+b) - \frac{2}{5}(a-b) \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -\frac{9}{5}a - \frac{6}{5}b \\ -\frac{12}{5}a - \frac{3}{5}b \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{5}a - \frac{2}{5}b \\ -\frac{4}{5}a - \frac{1}{5}b \end{bmatrix}$$

Thus we find that the general sol$^{\underline{n}}$ is

$$\boxed{\vec{x}(t) = c_1\begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix} + c_2\begin{bmatrix} e^{-5t} \\ -2e^{-5t} \end{bmatrix} + \begin{bmatrix} -\frac{3}{5}a - \frac{2}{5}b \\ -\frac{4}{5}a - \frac{1}{5}b \end{bmatrix}}$$

As $t \to \infty$ the $\vec{x}_1$ sol$^{\underline{n}}$ blows-up, it is a run-away arms race.

UNLESS $C_1 = 0$

PROBLEM 8 Continued

We were given that $x(0) = 1$ and $y(0) = 4$ these initial conditions will allow us to pin down $c_1$ & $c_2$,

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} -\frac{3}{5}a - \frac{2}{5}b \\ -\frac{4}{5}a - \frac{1}{5}b \end{bmatrix}$$

$$\begin{array}{rl} & \left( \begin{aligned} 1 &= c_1 + c_2 - \tfrac{3}{5}a - \tfrac{2}{5}b \\ 4 &= c_1 - 2c_2 - \tfrac{4}{5}a - \tfrac{1}{5}b \end{aligned} \right) \\ - & \\ \hline & -3 = 3c_2 + \tfrac{1}{5}a - \tfrac{1}{5}b \end{array} \longrightarrow \begin{array}{rl} & \left( \begin{aligned} 2 &= 2c_1 + 2c_2 - \tfrac{6}{5}a - \tfrac{4}{5}b \\ 4 &= c_1 - 2c_2 - \tfrac{4}{5}a - \tfrac{1}{5}b \end{aligned} \right) \\ + & \\ \hline & 6 = 3c_1 - \tfrac{10}{5}a - \tfrac{5}{5}b \end{array}$$

$$\begin{aligned} -15 &= 15c_2 + a - b \\ 15c_2 &= b - a - 15 \\ c_2 &= \tfrac{1}{15}(b - a - 15) \end{aligned} \qquad \begin{aligned} 3c_1 &= 6 + 2a + b \\ c_1 &= \tfrac{1}{3}(6 + 2a + b) \end{aligned}$$

Thus in total we find,

$$\boxed{\vec{x}(t) = \tfrac{1}{3}(6 + 2a + b)\begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix} + \tfrac{1}{15}(b - a - 15)\begin{bmatrix} e^{-5t} \\ -2e^{-5t} \end{bmatrix} + \tfrac{1}{5}\begin{bmatrix} -3a - 2b \\ -4a - b \end{bmatrix}}$$

There are two possibilities as $t \longrightarrow \infty$,

(i) $6 + 2a + b \neq 0$ then $\vec{x}(t) \longrightarrow \vec{\infty}$, RUN AWAY ARMS RACE

(ii) $6 + 2a + b = 0$ then $\vec{x}(t) \longrightarrow \frac{1}{5}\begin{bmatrix} -3a - 2b \\ -4a - b \end{bmatrix}$ as $t \to \infty$

Moreover, $b = -6 - 2a$ so $-3a - 2b = -3a + 12 + 4a$ and $-4a - b = -4a + 6 + 2a = 6 - 2a$. Thus

as $t \to \infty$, $\vec{x}(t) \longrightarrow \frac{1}{5}\begin{bmatrix} a + 12 \\ 6 - 2a \end{bmatrix}$ so the ARMS RACE stabelizes.

- Of course the model is crazy trust is <u>not</u> a constant!

$\boxed{\text{PROBLEM 9}}$ Find eigenvalues & eigenvectors + generalized eigen vectors for the matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ then solve $\vec{x}' = A\vec{x}$.

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$= -\lambda\left[ (1-\lambda)(1-\lambda) - 2(0) \right] - 0 + 1\left[ 0 \cdot 0 - (1-\lambda)\cdot 0 \right]$$

$$= -\lambda(1-\lambda)^2 = 0$$

$$\underbrace{\lambda_1 = 0 \;,\; \lambda_2 = 1}$$

Find eigenvectors for $\lambda_1$ and $\lambda_2$ say $\vec{u}_1$ and $\vec{u}_2$ as usual,

$$\vec{0} = A\vec{u}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \rightarrow \begin{matrix} w = 0 \\ v + 2w = 0 \\ \text{choose } u = 1 \end{matrix} \rightarrow \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{0} = (A - I)\vec{u}_2 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \rightarrow \begin{matrix} -u + w = 0 \\ 2w = 0 \\ \text{Choose } V = 1 \end{matrix} \rightarrow \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We have just two sol$^{n}$'s from these, namely $\vec{x}_1 = e^{0 \cdot t}\vec{u}_1$ and $\vec{x}_2 = e^{t}\vec{u}_2$ but we need three. The other comes from the generalized eigenvector $\vec{u}_3$. We use the "chain" and insist $(A - I)\vec{u}_3 = \vec{u}_2$ for many reasons,

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} -u + w = 0 \\ 2w = 1 \\ \text{choose } V = 0. \\ \text{and } w = 1/2 \\ u = w = \frac{1}{2}. \end{matrix} \rightarrow \vec{u}_3 = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

It is easy to verify that $\vec{u}_3$ is a generalized eigen vector of order two,

$$(A - I)^2 \vec{u}_3 = (A - I)(A - I)\vec{u}_3 = (A - I)\vec{u}_2 = 0.$$

now lets find the third sol$^{e}$ via the matrix exp.

PROBLEM 9

The fundamental identity we derived (w/o much work)

$$\begin{aligned}\vec{x}_3 = e^{At}\vec{u}_3 &= e^{t}\left(I + t(A-I) + \frac{t^2}{2}(A-I)^2 + \cdots\right)\vec{u}_3 \\ &= e^{t}\left(I\vec{u}_3 + t(A-I)\vec{u}_3 + \underbrace{\frac{t^2}{2}(A-I)^2\vec{u}_3 + \cdots}_{\text{these vanish thanks to the construction of } \vec{u}_3\text{ !}}\right. \\ &= e^{t}\left(\vec{u}_3 + t\vec{u}_2\right)\end{aligned}$$

digression (worth understanding)

The point here is that we need to use generalized eigenvectors to unravel the matrix exponential. It is not hard to prove $e^{At}$ is a fundamental matrix for $x' = Ax$, however it is hard to calculate $e^{At}$ w/o eigenvectors & generalized eigenvectors. I did not present $\vec{x}_1$ & $\vec{x}_2$ as products of the matrix exponential construction because it's uneeded to calculate them, however they stem from $e^{At}$ just the same, observe

$$\vec{x}_1 = e^{At}\vec{u}_1 = e^{0\cdot t}(I + tA + \cdots)\vec{u}_1 = I\vec{u}_1 + t\cancelto{0}{A\vec{u}_1} = \vec{u}_1$$

$$\vec{x}_2 = e^{At}\vec{u}_2 = e^{t}(I + t(A-I) + \cdots)\vec{u}_2 = e^{t}\vec{u}_2 + t\cancelto{0}{(A-I)\vec{u}_2} = e^{t}\vec{u}_2$$

see generally $\vec{x} = e^{At}\vec{u} = e^{\lambda t}\vec{u}$ when $(A-\lambda I)\vec{u} = 0$.

The general sol$^{\text{n}}$ is simply,

$$\boxed{\vec{x} = c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 e^{t}\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3 e^{t}\begin{bmatrix}1/2\\t\\1/2\end{bmatrix}}$$

**PROBLEM 10** We are given $\vec{u}_1, \vec{u}_2 \neq 0$ and LI with $(A-I)\vec{u}_1 = (A-I)\vec{u}_2 = 0$

$$\vec{x}_1 = e^{t}\vec{u}_1 \quad \text{and} \quad \vec{x}_2 = e^{t}\vec{u}_2$$

Using the usual eigenvector idea. Next consider $\vec{u}_3, \vec{u}_4, \vec{u}_5 \neq 0$ such that $(A-3I)\vec{u}_3 = 0$, $(A-3I)\vec{u}_4 = \vec{u}_3$ and $(A-3I)\vec{u}_5 = \vec{u}_4$.

$$\vec{x}_3 = e^{At}\vec{u}_3 = e^{3t}(I\vec{u}_3 + t(A-3I)\vec{u}_3 + \cdots) = e^{3t}\vec{u}_3.$$

$$\vec{x}_4 = e^{At}\vec{u}_4 = e^{3t}(I\vec{u}_4 + t(A-3I)\vec{u}_4 + \cdots) = e^{3t}(\vec{u}_4 + t\vec{u}_3).$$

$$\begin{aligned}\vec{x}_5 = e^{At}\vec{u}_5 &= e^{3t}(I\vec{u}_5 + t(A-3I)\vec{u}_5 + \tfrac{t^2}{2}(A-3I)^2\vec{u}_5 + \cdots) \\ &= e^{3t}(\vec{u}_5 + t\vec{u}_4 + \tfrac{t^2}{2}(A-3I)\vec{u}_4 + \cdots) \\ &= e^{3t}(\vec{u}_5 + t\vec{u}_4 + \tfrac{t^2}{2}\vec{u}_3).\end{aligned}$$

<u>Remark</u>: If this comes up on the test I expect you to reproduce the arguments from $e^{At}\vec{u}$ to the end, with the exception of $\vec{u}$ being a plain-old eigenvector. You are free to remember the pattern, but I want you to also understand where it comes from.

Next we have $\vec{u}_6 = \vec{a}_6 + i\vec{b}_6$ and $\vec{u}_7 = \vec{a}_7 + i\vec{b}_7$ with $(A-iI)\vec{u}_6 = 0$ and $(A-iI)\vec{u}_7 = \vec{u}_6$. To begin I'll find complex sol$^{n}$'s $\vec{W}_1$ and $\vec{W}_2$ (just a name)

$$\vec{W}_1 = e^{At}\vec{u}_6 = e^{it}(I\vec{u}_6 + \cancelto{0}{t(A-iI)\vec{u}_6 + \cdots}) = e^{it}\vec{u}_6 = \vec{W}_1$$

$$\vec{W}_2 = e^{At}\vec{u}_7 = e^{it}(I\vec{u}_7 + t(A-iI)\vec{u}_7 + \cdots) = e^{it}(\vec{u}_7 + t\vec{u}_6)$$

then we use $\vec{x}_6 = \mathrm{Re}(\vec{w}_1)$, $\vec{x}_7 = \mathrm{Im}(\vec{w}_1)$, $\vec{x}_8 = \mathrm{Re}(\vec{W}_2)$, $\vec{x}_9 = \mathrm{Im}(\vec{w}_2)$ which yield ↓

## PROBLEM 10 Continued

$$\vec{W}_1 = e^{it}\vec{u}_6 = (\cos t + i\sin t)(\vec{a}_6 + i\vec{b}_6)$$

$$= \underbrace{(\cos t)\vec{a}_6 - (\sin t)\vec{b}_6}_{\vec{X}_6} + i\,\underbrace{((\cos t)\vec{b}_6 + (\sin t)\vec{a}_6)}_{\vec{X}_7 = (\cos t)\vec{b}_6 + (\sin t)\vec{a}_6}$$

$$\vec{W}_2 = e^{it}(\vec{u}_7 + t\vec{u}_6)$$

$$= (\cos t + i\sin t)\left[\vec{a}_7 + t\vec{a}_6 + i(\vec{b}_7 + t\vec{b}_6)\right]$$

$$= \cos t\,(\vec{a}_7 + t\vec{a}_6) - \sin t\,(\vec{b}_7 + t\vec{b}_6) + \rightarrow$$

$$\hookrightarrow + i\left\{\cos t\,(\vec{b}_7 + t\vec{b}_6) + \sin t\,(\vec{a}_7 + t\vec{a}_6)\right\}$$

Therefore,

$$\vec{X}_8 = \cos t\,(\vec{a}_7 + t\vec{a}_6) - \sin t\,(\vec{b}_7 + t\vec{b}_6)$$

$$\vec{X}_9 = \cos t\,(\vec{b}_7 + t\vec{b}_6) + \sin t\,(\vec{a}_7 + t\vec{a}_6)$$

The general sol$^{\underline{n}}$ is formed by a linear combination of $\vec{X}_1, \vec{X}_2, \dots, \vec{X}_9$,

$$\underline{\vec{X} = c_1\vec{X}_1 + c_2\vec{X}_2 + c_3\vec{X}_3 + \dots + c_9\vec{X}_9}$$

Remark: I hope you can see that we can solve $\vec{X}' = A\vec{X}$ for <u>any</u> constant matrix A. We can solve any const. coeff linear system of ODEs, it's just a matter of calculation.