YOUR NAME HERE: ______________________________

MA 341, Introduction to Differential Equations

Instructor: James Cook
Practice Test : n-th order ODEs
Date: Tuesday, September 18, 2007

Directions: Show your work, if you doubt that you've shown enough detail then ask. If you need additional paper please ask. There are 106 pts to be earned, 6pts are bonus.

WARNING: this is a practice test, I will make the actual test shorter. Some problems will remain essentially the same but there is obviously too much for 75 minutes here.

1. (5pts) Assume that k is a given but unknown constant. Solve

$$\frac{dT}{dt} = k(T - 100)$$

your answer will include the constant k and some constant of integration.

2. (5 pts) Suppose a cup of coffee cools according to Newton's Law of Cooling. Further suppose that at $t = 0$ the coffee is at $T = 160$ then after one minute the coffee cools to $T = 140$. If the room the coffee is cooling in has an ambient temperature of 100 then find the temperature at time t. If my daughter Hannah likes to steal my coffee once it cools to 90 degrees do I need to worry about her sneaky plans to sip my coffee when I'm not looking ?

3. (10pts) Given that $v(0) = 10$ solve

$$\frac{dv}{dt} = -v^2$$

If v represents the velocity of a cat thrown horizontally with an initial velocity of 10 then does the cat ever come to rest ? Here you can interpret the term $-v^2$ as a frictional force. For a bonus point find if the position of the cat is bounded, use $v = dx/dt$.

4. (10pts) Solve

$$y + \frac{x}{2}\frac{dy}{dx} = \frac{1}{2x^3}$$

5. (10 pts) Find the solution of

$$e^x(y - x)dx + (1 + e^x)dy = 0$$

that passes through the point $(0, 0)$. Notice this is an exact equation.

6. (20 pts) Find the general solutions to the following differential equations, as usual we denote $D = d/dx$ and $y' = dy/dx$ etc...
   (a.) $y'' + 5y' + 6y = 0$
   (b.) $y'' + y' + y = 0$
   (c.) $(D^3 - 5D^2 + 6D)[y] = 0$
   (d.) $(D^2 + D + 1)^2[y] = 0$

7. (20 pts) Find the general solution of

$$y'' + 4y = x^2 + 5e^x$$

8. (15 pts) Find the general solution of

$$y'' + 5y' + 6y = cos(x) + 2sin(x)$$

9. (15 pts) Find the general solution of

$$y'' + y = 3x^2 + x + 1$$

10. (10 pts) Find the particular solution of

$$y'' + 4y = \tan(2x)$$

via the method of variation of parameters. Recall,

$$v_1 = \int \frac{-gy_2}{y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}}\,dx \qquad v_2 = \int \frac{gy_1}{y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}}\,dx$$

11. (6 pts) Find the general solution of

$$y'' + 4y = x^2 + 5e^x + \tan(2x)$$

please use the previous two problems for the sake of time.

12. (10 pts) Use the method of anihilators to find the correct form for the particular solution (don't find $A, B, C, ...$ just set it up) of the following differential equation, $D = d/dx$ as usual,

$$(D+1)(D^2+1)(D-3)^2[y](x) = e^{-x}$$

do the same for

$$(D+1)(D^2+1)(D-3)^2[y](x) = cos(x) + e^{-x}$$

and

$$(D+1)(D^2+1)(D-3)^2[y](x) = xcos(x) + sin(x)$$

13. (10pts) Use $e^{ix} = cos(x) + isin(x)$ to show that

$$sin^2(x) = \frac{1}{2}(1 - cos(2x))$$

it is useful to first recall the formula for cos(x) and sin(x) in terms of e^{ix}.

Solution to Practice Test, MA341 Fall 2007

#1, **#2** See problem 1 of TEST I from summer 2007

#3 $\frac{dv}{dt} = -v^2 \Rightarrow \int \frac{-dv}{-v^2} = \int dt \Rightarrow \frac{1}{v} = t + c$

then $v(0) = 10 \Rightarrow \frac{1}{10} = 0 + c \therefore \boxed{v = \frac{1}{t + 1/10}}$

I'll let you think about the bonus question.
Note $v \neq 0$ for any t so the cat never comes to rest.

#4 this is §2.3#10, see homework sol$^{\underline{n}}$'s pg. H10.

#5 $e^x(y-x)dx + (1+e^x)dy = 0$

$\frac{\partial F}{\partial x} = e^x y - xe^x \quad \& \quad \frac{\partial F}{\partial y} = 1 + e^x$

$F = \int \frac{\partial F}{\partial x} dx = \int (e^x y - xe^x) dx$

$= e^x y - \int xe^x$

$= e^x y - xe^x + e^x + C_1(y)$ (treated y as constant)

note: $u = x$, $dv = e^x dx$
$\int xe^x dx = xe^x - \int e^x dx$
$= xe^x - e^x$.

$\frac{\partial F}{\partial y} = e^x + \frac{\partial C_1}{\partial y} = 1 + e^x \therefore \frac{\partial C_1}{\partial y} = 1 \Rightarrow C_1 = y + C_2$

Thus our sol$^{\underline{n}}$'s are $e^x y - xe^x + e^x + y = k$. Since
$(0,0) \Rightarrow 0 - 0 + 1 + 0 = k \therefore \boxed{e^x y - xe^x + e^x + y = 1}$

#6 (a.) $y = c_1 e^{-2x} + c_2 e^{-3x}$

(b.) $\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2} \Rightarrow y = c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$.

(c.) $y = c_1 + c_2 e^{2x} + c_3 e^{3x}$

(d.) $y = c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right) + c_3 x e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_4 x e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$

(See problem 4 of summer 2007 for similar problem)

#7 on practice test

(20 pts) Find the general solution of

$$y'' + 4y = x^2 + 5e^x$$

$$(D^2+4)[y] = x^2 + 5e^x$$

$$\lambda^2 + 4 = 0 \therefore \lambda = \pm 2i \;\; \& \;\; \underline{y_h = c_1\cos(2x) + c_2\sin(2x)}.$$

I'll use annihilator method to form y_p, by experience ⤾

$$A = D^3(D-1)$$

then,

$$D^3(D-1)(D^2+4)[y] = 0$$

$$\Rightarrow y = \underbrace{c_1 + c_2x + c_3x^2 + c_4e^x}_{} + \underbrace{c_5\cos(2x) + c_6\sin(2x)}_{y_h}$$

$$y_p = A + Bx + Cx^2 + De^x$$

$$y_p' = B + 2Cx + De^x$$

$$y_p'' = 2C + De^x$$

Need $y_p'' + 4y_p = x^2 + 5e^x$,

$$2C + De^x + 4A + 4Bx + 4Cx^2 + 4De^x = x^2 + 5e^x$$

Equate coefficients,

$$\left.\begin{array}{ll} \underline{1|} & 2C + 4A = 0 \\ \underline{x|} & 4B = 0 \\ \underline{x^2|} & 4C = 1 \\ \underline{e^x|} & 5D = 5 \end{array}\right\}$$

$C = 1/4$
$A = -\frac{1}{2}C = -1/8$
$B = 0$
$\& \; D = 1$

$$\boxed{y = c_1\cos(2x) + c_2\sin(2x) - \frac{1}{8} + \frac{1}{4}x^2 + e^x}$$

#8 $y'' + 5y' + 6y = \cos(x) + 2\sin(x)$

$\lambda^2 + 5\lambda + 6 = (\lambda+3)(\lambda+2) = 0 \therefore \underline{y_h = c_1 e^{-3x} + c_2 e^{-2x}}$

Now find y_p via annihilator method (not req$^{\underline{d}}$ work, but w/o it do you know what to guess for y_p?)

$g = \cos(x) + 2\sin(x) \implies \lambda = \pm i \Rightarrow \lambda^2 + 1 = 0$

I can see $\cos(x)$ & $\sin(x)$ are sol$^{\underline{n}}$'s to $y'' + y = 0$.
thus I can see $A = D^2 + 1$ kills g.

$A[\cos(x) + 2\sin(x)] = (D^2+1)(\cos(x)) + 2(D^2+1)(\sin(x)) = 0.$

Our given nonhomogeneous eq$^{\underline{n}}$ was $(D^2 + 5D + 6)[y] = g$ now operate on both sides by $A = D^2 + 1$ to obtain,

$$(D^2+1)(D^2+5D+6)[y] = (D^2+1)[\cos(x) + 2\sin(x)] = 0$$

$$(D^2+1)(D+3)(D+2)[y] = 0.$$

$$\Rightarrow y = \underbrace{c_1\cos(x) + c_2\sin(x)}_{y_p = A\cos(x) + B\sin(x)} + \underbrace{c_3 e^{-3x} + c_4 e^{-2x}}_{y_h}$$

$$y_p' = -A\sin(x) + B\cos(x)$$

$$y_p'' = -A\cos(x) - B\sin(x)$$

Then substitute,

$$y_p'' + 5y_p' + 6y_p = \cos(x) + 2\sin(x)$$

$$-A\cos(x) - B\sin(x) + 5(-A\sin(x) + B\cos(x)) + 6(A\cos(x) + B\sin(x)) = g$$

$$\cos(x)[-A + 5B + 6A] + \sin(x)[-B - 5A + 6B] = \cos(x) + 2\sin(x)$$

Equate coefficients,

$$\left.\begin{aligned} \cos(x):&\ 5A - 5B = 1 \\ \sin(x):&\ 5B - 5A = 2 \end{aligned}\right\} \Rightarrow \begin{aligned} 10B &= 3 \\ 10A &= -1 \end{aligned} \Rightarrow \begin{aligned} B &= 3/10 \\ A &= -1/10 \end{aligned}$$

$$\therefore \boxed{y = c_1 e^{-3x} + c_2 e^{-2x} - \frac{1}{10}\cos(x) + \frac{3}{10}\sin(x)}$$

$\boxed{\#9}$ $y'' + y = 3x^2 + x + 1 = g$

$\lambda^2 + 1 = 0 \rightarrow \lambda = \pm i \rightarrow y_h = c_1 \cos(x) + c_2 \sin(x)$

Observe $A = D^3$ kills $3x^2 + x + 1$, $A[g] = 0$.
Next operate by A on $(D^2+1)[y] = 3x^2 + x + 1$,

$$D^3(D^2+1)[y] = D^3[3x^2+x+1] = 0$$

$$\lambda^3(\lambda^2+1) = 0 \rightarrow \lambda_1 = 0 = \lambda_2 = \lambda_3, \quad \lambda_{4,5} = \pm i$$

$$y = \underbrace{c_1 + c_2 x + c_3 x^2}_{y_p = A + Bx + Cx^2} + \underbrace{c_4 \cos(x) + c_5 \sin(x)}_{y_h}$$

$$y_p' = B + 2Cx$$

$$y_p'' = 2C$$

Then substitute into $L[y_p] = g$,

$$y_p'' + y_p = 3x^2 + x + 1$$

$$2C + A + Bx + Cx^2 = 3x^2 + x + 1$$

Equate Coefficients,

x^2: $C = 3$

x : $B = 1$

1 : $2C + A = 1 \longrightarrow A = 1 - 6 = -5$

Therefore, $y = y_h + y_p$ and,

$$\boxed{y = c_1 \cos(x) + c_2 \sin(x) - 5 + x + 3x^2}$$

(See problem 6 from summer 2007, there I just guess y_p, you can do that if you wish, I'm just demonstrating the method of annihilators yet again.)

$\boxed{\#10}$ see test I summer 2006 problem 7.

$\boxed{\#11}$ We found that in # 7,

$y'' + 4y = x^2 + 5e^x$ has $y_{p_1} = \frac{-1}{8} + \frac{1}{4}x^2 + e^x$

then from the sol$^{\underline{n}}$ to #10 mentioned above,

$y'' + 4y = \tan(2x)$ has $y_{p_2} = \frac{-1}{4}\cos(2x)\ln|\sec(2x) + \tan(2x)|$

given these two previous results we solve this problem by the principle of superposition,

$y_p = y_{p_1} + y_{p_2}$. Let me prove it works,

$$\begin{aligned} y_p'' + 4y_p &= (y_{p_1} + y_{p_2})'' + 4(y_{p_1} + y_{p_2}) \\ &= y_{p_1}'' + 4y_{p_1} + y_{p_2}'' + 4y_{p_2} \\ &= x^2 + 5e^x + \tan(2x). \end{aligned}$$

you can see from my proof why this only works for linear differential eq$^{\underline{n}}$'s, terms like $(y'')^2$ would mess-up the superposition principle. Physically speaking the superposition principle says we can treat forces one at a time then sum their influences to find the net result. (for linear physical systems only)

$\boxed{\#12}$ $(D+1)(D^2+1)(D-3)^2[y] = e^{-x} \rightarrow A = D+1$

$(D+1)^2(D^2+1)(D-3)^2[y] = 0$

$y = c_1e^{-x} + c_2xe^{-x} + c_3\cos(x) + c_4\sin(x) + c_5e^{3x} + c_6xe^{3x}$

these clearly will form y_h

$\boxed{y_p = Axe^{-x}}$

\#12) $(D+1)(D^2+1)(D-3)^2[y] = \cos(x) + e^{-x}$

$$A = (D^2+1)(D+1) \qquad \cos(x) \uparrow D^2+1 \qquad e^{-x} \uparrow D+1$$

$$(D^2+1)(D+1)(D+1)(D^2+1)(D-3)^2[y] = 0$$

$$(\lambda^2+1)^2(\lambda+1)^2(\lambda-3)^2 = 0$$

$\lambda = \pm i$ twice
$\lambda = -1$ twice
$\lambda = 3$ twice

$$y = c_1\cos(x) + c_2\sin(x) + c_3 x\cos(x) + c_4 x\sin(x) + c_5 e^{-x} + c_6 x e^{-x} + c_7 e^{3x} + c_8 x e^{3x}$$

this suggests,

$$\boxed{y_p = Ax\cos(x) + Bx\sin(x) + xe^{-x}}$$

Finally,

$$(D+1)(D^2+1)(D-3)^2 = x\cos(x) + \sin(x)$$

$x\cos(x) \uparrow (D^2+1)^2$ ← this also kills $\sin(x)$ so it's all we need for A.

$$(D^2+1)^2(D+1)(D^2+1)(D-3)^2 = 0$$

$$y = \underbrace{c_1 e^{3x} + c_2 x e^{3x} + c_3\cos(x) + c_4\sin(x) + c_5 e^{-x}}_{y_h} + \underbrace{c_6 x\cos(x) + c_7 x\sin(x) + c_8 x^2\cos(x) + c_9 x^2\sin(x)}$$

$$\boxed{y_p = Ax\cos(x) + Bx\sin(x) + Cx^2\cos(x) + Dx^2\sin(x)}$$

Remark: I probably write more than is entirely needed, if you find A and find y_p and give some indication you get the main idea you should recieve full credit.

$\boxed{\#13}$ Let me again state that the similar problem on test I will be identical in format.

$$\sin^2(x) = \frac{1}{2i}\left(e^{ix}-e^{-ix}\right)\frac{1}{2i}\left(e^{ix}-e^{-ix}\right)$$

$$= \frac{-1}{4}\left(e^{2ix} - 1 - 1 + e^{-2ix}\right)$$

$$= \frac{-1}{2}\,\frac{1}{2}\left(e^{2ix}+e^{-2ix}\right) - \frac{1}{4}(-2)$$

$$= \frac{-1}{2}\cos(2x) + \frac{1}{2}$$

$$= \frac{1}{2}\left(1 - \cos(2x)\right).$$

See Reg$^{\underline{d}}$ hwk #10 for similar problem. Also pg 5i - 6i in my lecture notes. (intro_to_complex.pdf)