Instructor: James Cook
Practice Test II: Systems of ODEs

WARNING: this is a practice test, I will make the actual test shorter. Some problems will remain essentially the same but there is obviously too much for 75 minutes here. I'll begin with the possible proof problems, I will give the formula for x_p if it is needed, however I would like for you to remember the formulas for the real solutions contained within the complex solution. I also expect you to understand how to solve $\vec{x}' = A\vec{x}$ using the matrix exponential, I will give you the formula

$$e^{At} = e^{\lambda t}(I + t(A-\lambda I) + \frac{1}{2}t^2(A-\lambda I)^2 + \frac{1}{3!}t^3(A-\lambda I)^3 + \cdots)$$

however, I expect you to know what to do with it. The format of the test will be three problems like the 29pt problems, 1 of the proof problems just as they are stated here, and two more challanging 4pt problems. There will be 105pts weighted by 100, that is 5 bonus pts possible. It is especially important that you explain steps on this test.

1. (10pts) Show that $\vec{x} = e^{\lambda t}\vec{u}$ is a nonzero solution to $\vec{x}' = A\vec{x}$ if we require that λ and $\vec{u}$ are constant with $det(A-\lambda I) = 0$ and $(A-\lambda I)\vec{u} = 0$. You will need to use a theorem from linear algebra which states that $B\vec{u} = 0$ has more than one solution only if $det(B) = 0$.

2. (10pts) Show that if $\vec{w} = Re(\vec{w}) + iIm(\vec{w})$ is a solution to $\vec{w}' = A\vec{w}$ then both $Re(\vec{w})$ and $Im(\vec{w})$ are also solutions.

3. (10pts) Show that if $\lambda = \alpha + i\beta$ and $\vec{u} = \vec{a} + i\vec{b}$ then

$$Re(e^{\lambda t}\vec{u}) = e^{\alpha t}\cos(\beta t)\vec{a} - e^{\alpha t}\sin(\beta t)\vec{b} \qquad and \qquad Im(e^{\lambda t}\vec{u}) = e^{\alpha t}\sin(\beta t)\vec{a} + e^{\alpha t}\cos(\beta t)\vec{b}.$$

Notice that we then have found how to extract two real solutions from the complex solution. I should mention that I assume here that $\alpha, \beta, \vec{a}, \vec{b}$ are all real, they have no $i = \sqrt{-1}$.

4. (10 pts) Show that $\vec{x_p} = X\vec{v}$ is a solution to $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$ if,

$$\vec{x_p}(t) = X(t)\int X^{-1}(t)\vec{f}(t)dt.$$

Here we assume X is a fundamental matrix for the system.

5. (10pts) Show that the matrix exponential is a fundamental matrix. That is show that e^{At} is invertible and it is a solution matrix for $\vec{x}' = A\vec{x}$.

6. (29pts) Rewrite the following system of differential equations in matrix normal form

$$x' = 2x - y \qquad\qquad y' = x + 2y.$$

Now find the general solution using our eigenvalue/eigenvector technique. Finally find the solution with $x(0) = 0$ and $y(0) = 1$ and write out the formulas for $x(t)$ and $y(t)$ seperately.

7. (29pts) Find the eigenvalues and generalized eigenvectors of the matrix below

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$$

then find the general solution to $\frac{d\vec{x}}{dt} = A\vec{x}$.

8. (29pts) Find the eigenvalues and generalized eigenvectors of the matrix below

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix}$$

then find the general solution to $\frac{d\vec{x}}{dt} = A\vec{x}$.

9. (29pts) Find the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ given that

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}$$

10. (29pts) Find the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ given that

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

11. (29pts) Find the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ given that

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

12. (29pts) Solve $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$ given that

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \vec{f} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

13. (29pts) Suppose that A is a 5x5 matrix such that $det(A-\lambda I) = (\lambda-1)^2(\lambda-3)^3$. Furthermore, suppose that

$$(A - I)\vec{u_1} = 0 \qquad and \qquad (A - I)\vec{u_2} = 0$$

where $\vec{u_1}, \vec{u_2}$ are nontrivial and linearly independent. Next suppose that,

$$(A - 3I)\vec{u_3} = 0 \qquad and \qquad (A - 3I)\vec{u_4} = \vec{u_3} \qquad and \qquad (A - 3I)\vec{u_5} = \vec{u_4}$$

where $\vec{u_3}, \vec{u_4}, \vec{u_5}$ are all nontrivial. Given all this data calculate the general solution to $\frac{d\vec{x}}{dt} = A\vec{x}$ in terms of the given vectors. You may use the formula on the board without proof. However, you should certainly show your work.

PRACTICE TEST II SOLUTION, FALL 2007, DIFFERENTIAL EQ^{n}S

PROBLEM ONE We need to make certain $\vec{x} = e^{\lambda t}\vec{u}$ solves $\vec{x}' = A\vec{x}$. We also assume that $\vec{x}$ is a nontrivial sol^n and $\lambda, \vec{u}$ constant.

$$\frac{d}{dt}\left(e^{\lambda t}\vec{u}\right) = \lambda e^{\lambda t}\vec{u} = Ae^{\lambda t}\vec{u}$$
$$\Rightarrow \lambda\vec{u} = A\vec{u}$$
$$\Rightarrow \boxed{(A - \lambda I)\vec{u} = 0.}$$

Now we note $\vec{u} = 0$ solves $(A - \lambda I)\vec{u} = 0$ but we are looking for $\vec{x} \neq 0 \Rightarrow \vec{u} \neq 0$. So there are at least two sol^n's to $(A - \lambda I)\vec{u} = 0$ hence by the Th^m from linear algebra $\boxed{\det(A - \lambda I) = 0}$

Remark: You might worry my argument does not make sense when $\lambda \in \mathbb{C}$, however it does for reasons I have elaborated on in the notes. I don't require you to include those details here.

PROBLEM TWO Let $\vec{w} = Re(\vec{w}) + i\,Im(\vec{w})$ with $\vec{w}' = A\vec{w}$ then

$$\vec{w}' = \frac{d}{dt}\left(Re(\vec{w}) + i\,Im(\vec{w})\right)$$
$$= \frac{d}{dt}\left(Re(\vec{w})\right) + i\frac{d}{dt}\left(Im(\vec{w})\right) \quad \text{— I}$$

$$A\vec{w} = A\left(Re(\vec{w}) + i\,Im(\vec{w})\right)$$
$$= A\,Re(\vec{w}) + i\,A\,Im(\vec{w}) \quad \text{— II}$$

we find two real equations from equating I and II

$$\frac{d}{dt}\left(Re(\vec{w})\right) = A\,Re(\vec{w})$$
$$\frac{d}{dt}\left(Im(\vec{w})\right) = A\,Im(\vec{w})$$

$Re(\vec{w})$ $Im(\vec{w})$

Thus given a complex sol^n $\vec{w}$ we get two real sol^{n}s.

$\boxed{\text{PROBLEM 3}}$ Let $\lambda = \alpha + i\beta$ and $\vec{u} = \vec{a} + i\vec{b}$ for real $\alpha, \beta, \vec{a}, \vec{b}$.

$$\begin{aligned}
e^{\lambda t}\vec{u} &= e^{(\alpha+i\beta)t}\,\vec{u} \\
&= e^{\alpha t} e^{i\beta t}\,\vec{u} \\
&= e^{\alpha t}(\cos\beta t + i\sin\beta t)(\vec{a} + i\vec{b}) \\
&= e^{\alpha t}\left[(\cos\beta t)\vec{a} - (\sin\beta t)\vec{b}\right] + i e^{\alpha t}\left[(\sin\beta t)\vec{a} + (\cos\beta t)\vec{b}\right]
\end{aligned}$$

Then we can read off the eq^{n} above,

$$\boxed{\begin{aligned}
Re(\vec{w}) &= e^{\alpha t}\left[(\cos\beta t)\vec{a} - (\sin\beta t)\vec{b}\right] \\
Im(\vec{w}) &= e^{\alpha t}\left[(\sin\beta t)\vec{a} + (\cos\beta t)\vec{b}\right]
\end{aligned}}$$

<u>Remark</u>: These formulas together with the proof of two provide the logical foundation for why we do what we do in the case of a complex eigenvalue.

$\boxed{\text{PROBLEM 4}}$ We are given $\Phi' = A\Phi$, $\det(\Phi) \neq 0$ since Φ is a fundamental matrix. We need $\vec{x}_p' = A\vec{x}_p + \vec{f}$ for the given guess $\vec{x}_p = \Phi\vec{v}$. Let's find the condition on $\vec{v}$,

$$\vec{x}_p' = (\Phi\vec{v})' = \underbrace{\frac{d\Phi}{dt}}_{A\Phi\vec{v}}\vec{v} + \Phi\frac{d\vec{v}}{dt} = A\vec{x}_p + \vec{f} = A\Phi\vec{v} + \vec{f}$$

$$\Rightarrow \Phi\frac{d\vec{v}}{dt} = \vec{f}$$

$$\Rightarrow \Phi^{-1}\Phi\frac{d\vec{v}}{dt} = \Phi^{-1}\vec{f}$$

$$\Rightarrow \frac{d\vec{v}}{dt} = \Phi^{-1}(t)\vec{f}(t)$$

$$\Rightarrow \vec{v}(t) = \int \Phi^{-1}(t)\vec{f}(t)\,dt$$ using the fundamental Th^{m} of Calculus.

$$\Rightarrow \boxed{\vec{x}_p = \Phi\int\Phi^{-1}\vec{f}\,dt}$$

PROBLEM 5 To show $\Phi = e^{At}$ is a fundamental matrix for $\vec{x}' = A\vec{x}$ need to exhibit two things (i) Φ^{-1} exists, (ii) $\Phi' = A\Phi$.

(i) Observe that $(e^{At})^{-1} = e^{-At}$ since

$$e^{At}e^{-At} = e^{At-At} = e^0 = I.$$

Where we knew $e^{A_1}e^{A_2} = e^{A_1+A_2}$ since $A_1 = At$ and $A_2 = -At$ and clearly $A_1A_2 = A_2A_1 = -A^2t^2$.

(ii)
$$\begin{aligned}
\Phi' &= \frac{d}{dt}\left(e^{At}\right) \\
&= \frac{d}{dt}\left(I + At + \tfrac{1}{2}t^2A^2 + \tfrac{1}{3!}t^3A^3 + \tfrac{1}{4!}t^4A^4 + \cdots\right) \\
&= A + tA^2 + \tfrac{1}{2}t^2A^3 + \tfrac{1}{3!}t^3A^4 + \cdots \\
&= A\left(I + tA + \tfrac{1}{2}t^2A^2 + \tfrac{1}{3!}t^3A^3 + \cdots\right) \\
&= Ae^{At} \\
&= A\Phi, \quad \therefore \underline{e^{At} \text{ is a sol}^{\underline{n}} \text{ matrix}}.
\end{aligned}$$

notice: $\frac{4}{4!} = \frac{\cancel{4}}{\cancel{4}\cdot 3\cdot 2\cdot 1} = \frac{1}{3\cdot 2\cdot 1} = \frac{1}{3!}$

Remark: In my notes I used $\det(e^A) = e^{\operatorname{trace}(A)}$ to argue that e^{At} is nonsingular, the argument I give here is easier since $\det(e^A) = e^{\operatorname{trace}(A)}$ is a deep identity. In fact it bridges the gap between "Lie Algebras" and "Lie Groups". The At is in the algebra while e^{At} is in the group. Ask me if you'd like to know more.

PROBLEM 6 This problem has the full treatment from beginning to end, usually we just do the middle part.

$$\begin{aligned} x' &= 2x - y \\ y' &= x + 2y \end{aligned} \qquad \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}_{\text{matrix normal form}}$$

$\boxed{\text{PROBLEM 6}}$

$$\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 + 1 = \lambda^2 - 4\lambda + 5 = 0$$

We find complex eigenvalues, $\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

Choose $\lambda = 2+i$ to be clear. Find eigenvector,

$$0 = (A-(2+i)I)\vec{u} = \begin{bmatrix} 2-2-i & -1 \\ 1 & 2-2-i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$-iu - v = 0 \longrightarrow v = -iu, \quad \text{let } u=1, \quad \vec{u} = \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

We see $\lambda = 2+i$ so $\alpha = 2$ & $\beta = 1$ and also $\vec{u} = \vec{a} + i\vec{b}$, $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$,

Using (or remembering depending on the test) PROBLEM 3,

$$\underbrace{\vec{x} = c_1 e^{2t}\left(\cos t\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) + c_2 e^{2t}\left(\sin t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)}_{\text{the general sol}^{\underline{n}}}$$

We were given $x(0) = 0$ and $y(0) = 1$

$$\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 \\ -c_2 \end{bmatrix} \quad \left(\text{used } \begin{matrix} \cos(0) = 1 \\ \sin(0) = 0 \end{matrix}\right)$$

Therefore, $c_1 = 0$ and $c_2 = -1$.

$$\vec{x}(t) = -e^{2t}\left(\sin t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -e^{2t}\sin t \\ e^{2t}\cos t \end{bmatrix}$$

Recall we constructed $\vec{x}(t)$ from $x(t)$ and $y(t)$ to begin with and $\vec{x}(t) = [x(t), y(t)]^T$ thus read off the two eq$^{\underline{n}}$'s,

$$\boxed{\begin{aligned} x(t) &= -e^{2t}\sin t \\ y(t) &= e^{2t}\cos t \end{aligned}}$$

Remark: usually it is ok to stop at $\vec{x}(t)$. The instructions here said otherwise.

PROBLEM 7 (This is problem 1 of TEST III of summer 2007. See that sol$^{\underline{n}}$ for details)

It can be shown that

$$\det(A-\lambda I) = (1-\lambda)(\lambda-5)(\lambda-2) = 0 \quad \therefore \underline{\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 5}$$

Then we can calculate __eigenvectors__ $\vec{u}_1 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

Therefore the __general__ sol$^{\underline{n}}$ is,

$$\boxed{\vec{x}(t) = c_1 e^{t}\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{5t}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}$$

__Remark__: here we needed only the most basic kind of generalized eigenvector, the plain-old eigenvector. We could have anticipated that we would not need higher order $\left((A-\lambda I)^k \vec{u} = 0 \quad k \geq 2 \right)$ gen. eigenvectors because our eigenvalues were __distinct__.

PROBLEM 8

$$\det(A-\lambda I) = \det\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -1 & -3 & -3-\lambda \end{bmatrix} = -\lambda(-\lambda(-3-\lambda)+3) - 1$$

$$= -\lambda(3\lambda + \lambda^2 + 3) - 1$$

$$= -\lambda^3 - 3\lambda^2 - 3\lambda - 1$$

$$= -(\lambda^3 + 3\lambda^2 + 3\lambda + 1) \longrightarrow$$

$$= -(\lambda+1)^3 \longleftarrow$$

1
1 1
1 2 1
1 3 3 1

We have $\underline{\lambda = -1 \text{ three times}}$.

__Remark__: You have calculator, the graph reveals much if you don't see the algebra right away.

PROBLEM 8 Find eigenvector $\vec{u}_1$ to begin with,

$$0=(A+I)\vec{u}_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -3 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$u+v=0 \quad \therefore \underline{v=-u}$

$v+w=0 \quad \therefore \underline{w=-v=u}.$

let $u=1 \Rightarrow \underline{\vec{u}_1 = [1,-1,1]^T}$

Apparently there is only one eigenvector here, we need to find three vectors in total. Let's look for $\vec{u}_2$ with $(A+I)\vec{u}_2 = \vec{u}_1$,

$$(A+I)\vec{u}_2 = \vec{u}_1 \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -3 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$u+v=1 \qquad v=1-u.$

$v+w=-1 \qquad w=-1-v=-1-(1-u)=-2+u$

Let $u=1$ we find. $\underline{\vec{u}_2 = [1,0,-1]^T}$

Next look for $\vec{u}_3$ with $(A+I)\vec{u}_3 = \vec{u}_2$,

$$(A+I)\vec{u}_3 = \vec{u}_2 \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -3 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$u+v=1 \qquad v=1-u$

$v+w=0 \qquad w=-v=-(1-u)=u-1$

Let $u=1$ then we find $\underline{\vec{u}_3 = [1,0,0]}$

When we have generalized eigenvectors we use the identity $e^{At} = e^{\lambda t}(I + t(A-\lambda I) + \cdots)$ to find sol^n's, here $\underline{\lambda=-1}$.
All of the sol^n's have the form $e^{At}\vec{u}$,

$$\vec{x}_1 = e^{At}\vec{u}_1 = e^{-t}(\vec{u}_1 + t(A+I)\vec{u}_1 + \cdots) = e^{-t}\vec{u}_1$$

(well we knew this w/o e^{At} no surprise here.)

for more interesting cases,

$\boxed{\text{PROBLEM 8}}$ Already did hard part, now we are using e^{At} to assemble the sol$^{\underline{n}}$'s,

we made $(A+I)\vec{u}_2 = \vec{u}_1$.

$$\vec{x}_2 = e^{At}\vec{u}_2 = e^{-t}\left(\vec{u}_2 + t\underbrace{(A+I)\vec{u}_2} + \frac{t^2}{2}(A+I)(A+I)\vec{u}_2 + \cdots\right)$$
$$= e^{-t}\left(\vec{u}_2 + t\vec{u}_1 + \frac{t^2}{2}(A+I)\vec{u}_1 + \cdots\right)$$
$$= e^{-t}\left(\vec{u}_2 + t\vec{u}_1\right).$$

$$\vec{x}_3 = e^{At}\vec{u}_3 = e^{-t}\left(\vec{u}_3 + t(A+I)\vec{u}_3 + \frac{t^2}{2}(A+I)^2\vec{u}_3 + \frac{t^3}{3!}(A+I)^3\vec{u}_3 + \cdots\right)$$
$$= e^{-t}\left(\vec{u}_3 + t\vec{u}_2 + \frac{1}{2}t^2(A+I)\vec{u}_2 + \frac{t^3}{3!}(A+I)^2\vec{u}_2 + \cdots\right.$$
$$= e^{-t}\left(\vec{u}_3 + t\vec{u}_2 + \frac{1}{2}t^2\vec{u}_1 + \frac{t^3}{3!}\underbrace{\cancel{(A+I)\vec{u}_1}}_{0} + \cdots\right)$$
$$= e^{-t}\left(\vec{u}_3 + t\vec{u}_2 + \frac{1}{2}t^2\vec{u}_1\right).$$

Remark: It is true that we could just as well choose any three linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and say $\vec{y}_1 = e^{At}\vec{v}_1$, $\vec{y}_2 = e^{At}\vec{v}_2$, $\vec{y}_3 = e^{At}\vec{v}_3$. Trouble is that we would have little chance of actually being able to find explicit formulas for $\vec{y}_1, \vec{y}_2, \vec{y}_3$. The fact that $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are generalized eigenvectors allows us to find nice formulas, The identity

$$e^{At} = e^{\lambda t}\left(I + t(A-\lambda I) + \frac{t^2}{2}(A-\lambda I)^2 + \frac{t^3}{3!}(A-\lambda I)^3 + \cdots\right)$$

is what makes all this possible. I put it on board during test.

Finally then the answer is,

$$\boxed{\vec{x}(t) = c_1 e^{-t}\begin{bmatrix}1\\-1\\1\end{bmatrix} + c_2 e^{-t}\left(\begin{bmatrix}1\\0\\-1\end{bmatrix} + t\begin{bmatrix}1\\-1\\1\end{bmatrix}\right) + c_3 e^{-t}\left(\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}0\\0\\-1\end{bmatrix} + \frac{t^2}{2}\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)}$$

PROBLEM 9 Solve $\vec{x}' = A\vec{x}$ where $A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -4 & 4-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0 \quad \therefore \underline{\lambda = 2 \text{ twice}}$$

Look for eigenvector $\vec{u}_1$,

$$0 = (A - 2I)\vec{u}_1 = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \qquad \begin{array}{l} -2u + v = 0 \\ v = 2u \\ \text{let } u = 1 \end{array} \quad \therefore \quad \underline{\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

Look for generalized e.v. $\vec{u}_2$,

$$\vec{u}_1 = (A - 2I)\vec{u}_2 = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \begin{array}{l} -2u + v = 1 \\ v = 1 + 2u \\ \text{let } u = 1 \end{array} \quad \therefore \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Calculate then

$$\begin{aligned} \vec{x}_2 = e^{At}\vec{u}_2 &= e^{2t}\left(\vec{u}_2 + t(A - 2I)\vec{u}_2 + \tfrac{t^2}{2}(A - 2I)(A - 2I)\vec{u}_2 + \cdots\right) \\ &= e^{2t}\left(\vec{u}_2 + t\vec{u}_1 + \tfrac{t^2}{2}\cancel{(A - 2I)\vec{u}_1}^{\to 0} + \cdots\right) \end{aligned}$$

So we find the general sol$^{\underline{n}}$,

$$\boxed{\vec{x}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{2t}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)}$$

__Remark__: I gave a minimal sol$^{\underline{n}}$ here, I'd like you to show at __least__ these details for full credit. You have been warned.

PROBLEM 10 $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ $\quad A^2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

So for this special (nilpotent) matrix we calculate

$$e^{At} = I + tA + \tfrac{1}{2}t^2 \underbrace{A^2}_{0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix}$$

$$\therefore \quad \boxed{\vec{x}(t) = c_1\begin{bmatrix} 1+t \\ t \end{bmatrix} + c_2\begin{bmatrix} -t \\ 1-t \end{bmatrix}}$$

__Remark__: we could also solve this like PROB. 9, $\underline{\lambda = 0 \text{ twice}}$.

PROBLEM 11 Solve $\vec{x}\,' = A\vec{x}$ for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0 \quad \therefore \underline{\lambda = \pm i}$$

Choose $\lambda = i$ to be clear. Find eigenvector,

$$0 = (A - iI)\vec{u} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} -iu + v = 0 \\ \therefore v = iu \\ \text{let } u = 1 \end{matrix} \qquad \vec{u} = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Identify that $\vec{u} = \vec{a} + i\vec{b}$ for $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ since clearly $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus, since $\alpha = 0$, $\beta = 1$

$$\boxed{\vec{x}(t) = c_1\left(\cos t\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + c_2\left(\sin t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos t\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)}$$

PROBLEM 12 From problem 11 we see

$$\Phi = [\vec{x}_1 \,|\, \vec{x}_2] = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \quad (\text{with } \det(\Phi) = \cos^2 t + \sin^2 t = 1.)$$

$$\Rightarrow \Phi^{-1} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

Recall $\vec{x}_p = \Phi \int \Phi^{-1} f\, dt$ so calculate,

$$\begin{aligned} \vec{x}_p &= \Phi \int \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} dt \\ &= \Phi \int \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} dt \\ &= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} \\ &= \begin{bmatrix} \cos t \sin t - \sin t \cos t \\ -\sin^2 t - \cos^2 t \end{bmatrix} \\ &= \underline{\begin{bmatrix} 0 \\ -1 \end{bmatrix}} \quad \therefore \quad \boxed{\vec{x}(t) = c_1\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}} \end{aligned}$$

$\boxed{\text{PROBLEM 13}}$ We were <u>given</u> that A is a 5×5 matrix with
$(A-I)\vec{u}_1 = 0$, $(A-I)\vec{u}_2 = 0$ $\vec{u}_1$ & $\vec{u}_2$ LI vectors.
$(A-3I)\vec{u}_3 = 0$, $(A-3I)\vec{u}_4 = \vec{u}_3$, $(A-3I)\vec{u}_5 = \vec{u}_4$, $\vec{u}_3, \vec{u}_4, \vec{u}_5 \neq 0$
The "formula on the board" is

$$e^{At} = e^{\lambda t}\left(I + t(A-\lambda I) + \frac{t^2}{2}(A-\lambda I)^2 + \frac{t^3}{3!}(A-\lambda I)^3 + \cdots\right)$$

For $\vec{u}_1, \vec{u}_2, \vec{u}_3$ we note $\vec{x}_1 = e^t\vec{u}_1$, $\vec{x}_2 = e^t\vec{u}_2$, $\vec{x}_3 = e^{3t}\vec{u}_3$, (logically follows from $\boxed{\text{PROBLEM 2}}$ or the formula plus $\boxed{\text{PROBLEM 5}}$.)

$$\begin{aligned}
\vec{x}_4 &= e^{At}\vec{u}_4 \\
&= e^{3t}\left(\vec{u}_4 + t(A-3I)\vec{u}_4 + \frac{t^2}{2}(A-3I)(A-3I)\vec{u}_4 + \cdots\right) \\
&= e^{3t}\left(\vec{u}_4 + t\,\vec{u}_3 + \frac{t^2}{2}\underbrace{\cancel{(A-3I)\vec{u}_3}}_{\to 0}\right)
\end{aligned}$$

$$\begin{aligned}
\vec{x}_5 &= e^{At}\vec{u}_5 \\
&= e^{3t}\left(\vec{u}_5 + t(A-3I)\vec{u}_5 + \frac{t^2}{2}(A-3I)^2\vec{u}_5 + \frac{t^3}{3!}(A-3I)^3\vec{u}_5 + \cdots\right) \\
&= e^{3t}\left(\vec{u}_5 + t\,\vec{u}_4 + \frac{t^2}{2}(A-3I)\vec{u}_4 + \frac{t^3}{3!}(A-3I)^2\vec{u}_4 + \cdots\right) \\
&= e^{3t}\left(\vec{u}_5 + t\,\vec{u}_4 + \frac{t^2}{2}\vec{u}_3 + \frac{t^3}{3!}\underbrace{\cancel{(A-3I)\vec{u}_3}}_{\to 0} + \cdots\right)
\end{aligned}$$

Thus the general sol$^{\underline{n}}$ is

$$\boxed{\begin{aligned}\vec{x}(t) = c_1e^t\vec{u}_1 + c_2e^t\vec{u}_2 + c_3e^{3t}\vec{u}_3 \\ + c_4e^{3t}(\vec{u}_4 + t\vec{u}_3) + c_5e^{3t}\left(\vec{u}_5 + t\vec{u}_4 + \frac{t^2}{2}\vec{u}_3\right)\end{aligned}}$$