YOUR NAME HERE:

MA 341, Introduction to Differential Equations

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Test II: Systems of linear ODEs
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Directions: Show your work, if you doubt that you've shown enough detail then ask. If you need additional paper please ask. There are 105 pts to be earned, 5pts are bonus. To begin let me remind you that

$$e^{At} = e^{\lambda t}(I + t(A - \lambda I) + \frac{1}{2}t^2(A - \lambda I)^2 + \frac{1}{3!}t^3(A - \lambda I)^3 + \cdots)$$

1. (10pts) Show that if $\lambda = \alpha + i\beta$ and $\vec{u} = \vec{a} + i\vec{b}$ then

$$Re(e^{\lambda t}\vec{u}) = e^{\alpha t}\cos(\beta t)\vec{a} - e^{\alpha t}\sin(\beta t)\vec{b} \qquad and \qquad Im(e^{\lambda t}\vec{u}) = e^{\alpha t}\sin(\beta t)\vec{a} + e^{\alpha t}\cos(\beta t)\vec{b}.$$

Notice that we then have found how to extract two real solutions from the complex solution. I should mention that I assume here that $\alpha, \beta, \vec{a}, \vec{b}$ are all real, they have no $i = \sqrt{-1}$.

2. **(29pts) Rewrite the following system of differential equations in matrix normal form**

$$x' = -x - 2y \qquad\qquad y' = 8x - y.$$

Now find the general solution using our eigenvalue/eigenvector technique. Finally find the solution with $x(0) = 0$ and $y(0) = 1$ and write out the formulas for $x(t)$ and $y(t)$ seperately.

3. (29pts) Find the eigenvalues, eigenvector and generalized eigenvector of the matrix below

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

then find the general solution to $\frac{d\vec{x}}{dt} = A\vec{x}$.

4. (29pts) Find the eigenvalues and eigenvectors of the matrix below

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

then find the general solution to $\frac{d\vec{x}}{dt} = A\vec{x}$.

5. (4pts) Consider the following matrices

$$X = \begin{pmatrix} e^t & 3e^{-2t} \\ -e^t & 0 \end{pmatrix} \qquad\qquad Y = \begin{pmatrix} e^t & -2e^t \\ -e^t & 2e^t \end{pmatrix}$$

Are these fundamental matrices? If so are they fundamental matrices of the same system ? **What is the system or systems that these matrices are solution matrices for? If possible find** the explicit form of A for the system(s) (where I have $\frac{d\vec{x}}{dt} = A\vec{x}$ in mind).

6. (4pts) Use the variation of parameters for a 2x2 system $\vec{x}' = A\vec{x} + \vec{f}$ to derive the formulas for variation of parameters for the 2nd order system $ay'' + by' + cy = g$. You may assume that a is nonzero. You will need to use the following formula,

$$\vec{x}_p(t) = X(t) \int X^{-1}(t)\vec{f}(t)dt.$$

also you should assume that y_1, y_2 are the fundamental solutions of $ay'' + by' + cy = g$. Finally I found it useful to do the usual exchange $x_1 = y$ and $x_2 = y'$.

SOLUTION TO TEST II ON SYSTEMS OF ODES, FALL 2007

PROBLEM 1 Let $\lambda = \alpha + i\beta$ and $\vec{u} = \vec{a} + i\vec{b}$ for $\alpha, \beta, \vec{a}, \vec{b}$ real then,

$$\begin{aligned}
e^{\lambda t}\vec{u} &= e^{(\alpha+i\beta)t}\vec{u} \\
&= e^{\alpha t}e^{i\beta t}\vec{u} \\
&= (e^{\alpha t}\cos\beta t + ie^{\alpha t}\sin\beta t)(\vec{a} + i\vec{b}) \\
&= (e^{\alpha t}\cos\beta t)\vec{a} - (e^{\alpha t}\sin\beta t)\vec{b} + i\{(e^{\alpha t}\sin\beta t)\vec{a} + (e^{\alpha t}\cos\beta t)\vec{b}\}
\end{aligned}$$

I used $i^2 = -1$ to get). Now we can read from the eq$^{\underline{n}}$ above that

$$\boxed{\begin{aligned}
Re(e^{\lambda t}\vec{u}) &= (e^{\alpha t}\cos\beta t)\vec{a} - (e^{\alpha t}\sin\beta t)\vec{b} \\
Im(e^{\lambda t}\vec{u}) &= (e^{\alpha t}\sin\beta t)\vec{a} + (e^{\alpha t}\cos\beta t)\vec{b}
\end{aligned}}$$

PROBLEM 2

$$\begin{aligned} x' &= -x - 2y \\ y' &= 8x - y \end{aligned} \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{bmatrix} -1 & -2 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{bmatrix} -1-\lambda & -2 \\ 8 & -1-\lambda \end{bmatrix} = (-1-\lambda)(-1-\lambda) + 16 = \lambda^2 + 2\lambda + 17$$

$$\lambda = \frac{-2 \pm \sqrt{4-68}}{2} = -1 \pm \frac{\sqrt{-64}}{2} = -1 \pm 4i$$ consider $\underline{\lambda = -1+4i}$ $(\alpha = -1, \beta = 4)$

Find Eigenvector with $\lambda = -1 + 4i$

$$(A - \lambda I)\vec{u} = \begin{bmatrix} -1-(-1+4i) & -2 \\ 8 & -1-(-1+4i) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -4i & -2 \\ 8 & -4i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$-4iu - 2v = 0$
$\Rightarrow \underline{v = -2iu}$.

Let $u = 1$ so $\vec{u} = \begin{bmatrix} 1 \\ -2i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ -2 \end{bmatrix}$

identify that $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$.

$\boxed{\text{PROBLEM 2 Continued}}$

The general sol^{n} is, (factoring out the $e^{\alpha t} = e^{-t}$ term)

$$\vec{x}(t) = c_1 e^{-t}\left(\cos 4t \begin{bmatrix}1\\0\end{bmatrix} - \sin 4t \begin{bmatrix}0\\-2\end{bmatrix}\right) + c_2 e^{-t}\left(\sin 4t \begin{bmatrix}1\\0\end{bmatrix} + \cos 4t \begin{bmatrix}0\\-2\end{bmatrix}\right)$$

Next apply initial conditions $x(0) = 0$ & $y(0) = 1$

$$\begin{bmatrix}0\\1\end{bmatrix} = c_1 e^{0}\left(\cos(0)\begin{bmatrix}1\\0\end{bmatrix}\right) + c_2 e^{0}\left(\cos(0)\begin{bmatrix}0\\-2\end{bmatrix}\right)$$

$\sin(0) = 0$ terms vanish.

$$\begin{bmatrix}0\\1\end{bmatrix} = c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\-2\end{bmatrix} = \begin{bmatrix}c_1\\-2c_2\end{bmatrix}$$

$\underline{c_1 = 0}$ & $-2c_2 = 1 \therefore \underline{c_2 = -1/2}$.

$$\vec{x}(t) = \frac{-1}{2} e^{-t}\left(\sin 4t \begin{bmatrix}1\\0\end{bmatrix} + \cos 4t \begin{bmatrix}0\\-2\end{bmatrix}\right)$$

$$= \begin{bmatrix}-\frac{1}{2}e^{-t}\sin 4t\\0\end{bmatrix} + \begin{bmatrix}0\\e^{-t}\cos 4t\end{bmatrix}$$

$$= \begin{bmatrix}-\frac{1}{2}e^{-t}\sin 4t\\e^{-t}\cos 4t\end{bmatrix} = \begin{bmatrix}x(t)\\y(t)\end{bmatrix}$$

$$\therefore \boxed{\begin{aligned} x(t) &= \frac{-1}{2}e^{-t}\sin 4t \\ y(t) &= e^{-t}\cos 4t \end{aligned}}$$

$\boxed{\text{PROBLEM 3}}$ Solve $\vec{x}' = A\vec{x}$ given $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{bmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$$

$\underline{\lambda = 1 \text{ twice}}$.

Find eigenvector $\vec{u}_1$,

$$0 = (A - I)\vec{u}_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$u + v = 0 \quad \therefore v = -u$

let $u = 1$ so $\underline{\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$.

Next find generalized eigenvector $\vec{u}_2$,

$$\vec{u}_1 = (A - I)\vec{u}_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$u + v = 1$

$v = 1 - u$

let $u = 0$

$\underline{\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$.

Observe that

$$(A - I)(A - I)\vec{u}_2 = (A - I)\vec{u}_1 = 0$$

$$\therefore \underline{(A - I)^2 \vec{u}_2 = 0}$$

Thus,

$$\begin{aligned} \vec{x}_2 &= e^{At}\vec{u}_2 \\ &= e^t\left(\vec{u}_2 + t(A - I)\vec{u}_2 + \cancel{\tfrac{t^2}{2}(A-I)^2\vec{u}_2 + \cdots}\right) \\ &= e^t(\vec{u}_2 + t\vec{u}_1) \end{aligned}$$

The general sol^{n} is

$$\boxed{\vec{x} = c_1 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^t\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)}.$$

PROBLEM 4 Solve $\vec{x}' = A\vec{x}$ for $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 3 & 2-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{bmatrix}$$

$$= (1-\lambda)(2-\lambda)(3-\lambda) \quad \therefore \ \underline{\lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 3}.$$

Find eigenvector $\vec{u}_1$ with $\lambda_1 = 1$,

$$(A - I)\vec{u}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3u + v = 0 \rightarrow \underline{v = -3u}$

$u + 2w = 0 \rightarrow \underline{w = -\frac{1}{2}u}$

let $u = 2$.

$$\vec{u}_1 = \begin{bmatrix} 2 \\ -6 \\ -1 \end{bmatrix}.$$

Next $\vec{u}_2$ with $\lambda_2 = 2$,

$$(A - 2I)\vec{u}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-u = 0$

$u + w = 0 \quad \therefore \ w = -u = 0$

notice v is free

choose $v = 1$.

$$\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Next $\vec{u}_3$ with $\lambda_3 = 3$,

$$(A - 3I)\vec{u}_3 = \begin{bmatrix} -2 & 0 & 0 \\ 3 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$-2u = 0 \rightarrow u = 0.$

$3u - v = 0 \rightarrow v = 3u = 0.$

notice w is free

choose $w = 1$

$$\vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus the general sol^{n},

$$\boxed{\vec{x}(t) = c_1 e^{t} \begin{bmatrix} 2 \\ -6 \\ -1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$\boxed{\text{PROBLEM 5}}$ Consider $\Sigma = \begin{bmatrix} e^t & 3e^{-2t} \\ -e^t & 0 \end{bmatrix}$ & $\Upsilon = \begin{bmatrix} e^t & -2e^t \\ -e^t & 2e^t \end{bmatrix}$

- Clearly Υ is <u>not</u> a fundamental matrix because $\det(\Upsilon) = 2e^te^t - 2e^te^t = 0$ so the columns of Υ are linearly dependent. In fact we can see that if $\Upsilon = [\vec{y}_1 | \vec{y}_2]$ then $\vec{y}_2 = -2\vec{y}_1$.

- Now $\det(\Sigma) = 3e^{-2t}e^t = 3e^{-t} \neq 0$ so Σ has linearly independent columns it can be a fundamental matrix. Also we see that if $\Sigma = [\vec{x}_1 | \vec{x}_2]$ clearly $\vec{x}_1 = \vec{y}_1 = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$ however $\vec{x}_2$ is only inside Σ.

So both Σ and Υ are apparently sol$^{\underline{n}}$ matrices for the same system. It is a system with solutions

$$\vec{x}_1 = e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{x}_2 = e^{-2t} \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

There are likely other ways to go on from here but I'll do it my way, Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$(A - I)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$a - 1 - b = 0 \qquad b = a - 1 = 1$$

$$c + 1 - d = 0 \qquad d = c + 1 = 1$$

$$(A - 2I)\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} a-2 & b \\ c & d-2 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3a - 6 = 0 \quad \therefore \boxed{a = 2}$$

$$3c = 0 \quad \therefore \boxed{c = 0}$$

These are sol$^{\underline{n}}$ matrices for $\vec{x}\,' = A\vec{x}$ where $\underline{A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}$

Problem 6

Notice that $ay'' + by' + cy = g$ can be rewritten as a system of two 1st order ODEs under the usual exchange of variables: $x_1 \equiv y$, $x_2 \equiv y'$ then

$$ay'' + by' + cy = g \longrightarrow ax_2' + bx_2 + cx_1 = g$$
$$\Rightarrow x_2' = \frac{-c}{a}x_1 - \frac{b}{a}x_2 + \frac{g}{a}$$

Also $x_1' = x_2$

Thus in matrix form we have $\vec{x}' = A\vec{x} + \vec{f}$
$\vec{x} = (x_1, x_2)^T$ and $A = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix}$ with $\vec{f} = \begin{bmatrix} 0 \\ g/a \end{bmatrix}$
Now we assume y_1 and y_2 were sol^{n}'s thus we have corresponding sol^{n}'s to the system namely

$$\vec{x}_1 = \begin{bmatrix} y_1 \\ y_1' \end{bmatrix} \quad \text{and } \vec{x}_2 = \begin{bmatrix} y_2 \\ y_2' \end{bmatrix}$$

We can form the fundamental matrix $\Phi = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$.
Calculate $\det(\Phi) = y_1 y_2' - y_2 y_1'$. Now we can use 2x2 variation of parameters,

$$\vec{x}_p = \Phi \int \Phi^{-1}\vec{f}\, dt = \Phi \int \frac{1}{\det(\Phi)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ g/a \end{bmatrix} dt$$

$$= \Phi \int \frac{1}{y_1y_2' - y_2y_1'} \begin{bmatrix} -y_2 g/a \\ y_1 g/a \end{bmatrix} dt$$

$$= \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} \int \frac{-y_2 g\, dt}{y_1y_2' - y_2y_1'} \\ \int \frac{y_1 g\, dt}{y_1y_2' - y_2y_1'} \end{bmatrix}$$

$$= \begin{bmatrix} y_1V_1 + y_2V_2 \\ y_1'V_1' + y_2'V_2' \end{bmatrix} = \begin{bmatrix} y_p \\ y_p' \end{bmatrix}$$

(we were trying to find these)

$$\boxed{V_1 = \int \frac{-y_2 g}{y_1y_2' - y_2y_1'} dt \qquad V_2 = \int \frac{y_1 g}{y_1y_2' - y_2y_1'} dt}$$

Where V_1 and V_2 are as we found before from a different argument.