INTEGRATING FACTOR METHOD

Sometimes we will encounter 1^{st} order DE_{q}^{n}'s which are not seperable with algebra alone. The question of how to make an arbitraily first order DE_{q}^{n} into a seperable DE_{q}^{n} is an interesting one (see Hydon's book if you're interested) Many 1^{st} order DE_{q}^{n}'s require great ingenuity to solve, however the following will not be to difficult,

$$\boxed{\frac{dY}{dx} + P(x)\,Y = Q(x)} \leftarrow \text{"standard form"}$$

This is a <u>linear</u> 1^{st} order ODE in standard form. We can make this eq^{n} seperable, in some sense, by multiplying by the <u>integrating factor</u> $\mu(x)$,

$$\boxed{\mu(x) \equiv \exp\left(\int P(x)\,dx\right)}$$

Lets see why this helps, multiply the DE_{q}^{n} by $\mu(x)$

$$\underbrace{\mu(x)\frac{dY}{dx} + \mu(x)P(x)\,Y}\; = \mu(x)\,Q(x)$$

$$\boxed{\frac{d}{dx}\big(\mu(x)\,Y\big) = \mu(x)\,Q(x)}$$

Using $(fg)' = f'g + fg'$ the product rule. And the observation

$$\frac{d}{dx}\big(\mu(x)\big) = \frac{d}{dx}\left(e^{\int P(x)dx}\right) = e^{\int P(x)dx}\frac{d}{dx}\int P(x)dx = \mu(x)\,P(x).$$

The identity just above is the reason for defining $\mu(x)$ as we did. Continuing from $(*)$ dropping x-dependence

$$d(\mu Y) = \mu Q\,dx$$

$$\mu Y = \int \mu Q\,dx \qquad : \text{ note } \mu(x) \neq 0 \text{ we can divide,}$$

$$\boxed{Y = \frac{1}{\mu}\int \mu Q\,dx}$$

this solves any linear ODE in standard form assuming $P(x)$, $Q(x)$ continuous.

E1 Find general sol^n to $\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos(x)$. We know we can solve it once it's put into standard form, so make it so,

$$\frac{dy}{dx} - \underbrace{\left(\frac{2}{x}\right)}_{P(x)} y = \underbrace{x^2\cos(x)}_{Q(x)}$$

$$\mu = \exp\left(\int -\frac{2}{x}\,dx\right) = \exp\left(-2\ln|x|\right) = \exp\left(\ln\left(\frac{1}{|x|^2}\right)\right) = \frac{1}{x^2}$$

Multiplying by μ yields,

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = \cos(x)$$

Nice, $\frac{d}{dx}\left(\frac{1}{x^2}y\right) = \cos(x) \Rightarrow \frac{y}{x^2} = \sin(x) + C$

$$\Rightarrow \boxed{y = x^2(C + \sin(x))}$$

Remark: see Fig. 2.5 for a picture of these sol^n's for various C.

E2 $y\frac{dx}{dy} + 2x = 5y^3$ (this one is a bit weird, we'll need to think of x as the dependent variable and y as the independent variable)

Standard Form $\rightarrow$ $\frac{dx}{dy} + \left(\frac{2}{y}\right)x = 5y^2$

$$\mu(y) = \exp\left(\int\left(\frac{2}{y}\right)dy\right) = \exp\left(2\ln|y|\right) = \exp\left(\ln|y|^2\right) = y^2$$

$$\underbrace{y^2\frac{dx}{dy} + 2yx}_{\frac{d}{dy}(y^2x)} = 5y^4 \quad \text{Multiplied by } \mu(y) = y^2.$$

$$\frac{d}{dy}\left(y^2x\right) = 5y^4 \Rightarrow y^2x = y^5 + C$$

$$\Rightarrow \boxed{x = y^3 + C/y^2}$$

Remark: the integrating factor method is rather simple if you can do the req^d integrations. In the event the integrals are incalculable we can still use the method numerically, but we'll stick to the cases where I can solve the integrals, I hope you can to ☺.

EXACT EQUATIONS

An exact equation arises as the total differential for some function of several variables $F(x,y)$,

$$\boxed{dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0} \quad \text{exact eq}^{n}.$$

This eq^{n} has sol^{n}'s which are level surfaces for $F(x,y)$. That means sol^{n}'s have the form $F(x,y) = c$, the total differential of a constant is zero so $dF = 0$ and it is indeed a sol^{n} to the exact eq^{n} as claimed. So if we <u>know</u> $F(x,y)$ solving $dF = 0$ is very easy. The question then is how do we know if a given DEq^{n} is exact, and if it is then what is F?

E1 $\dfrac{dy}{dx} = -\dfrac{2xy^2+1}{2x^2y}$ is there such a $F = F(x,y)$?

$$(2xy^2+1)dx + (2x^2y)dy \overset{(?)}{=} dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

We would need $\dfrac{\partial F}{\partial x} = 2xy^2+1$ and $\dfrac{\partial F}{\partial y} = 2x^2y$.
A little thought will reveal that $F = x^2y^2 + x$ will work. Thus the sol^{n}'s to the above DEq^{n} are simply,

$$\boxed{x^2y^2 + x = c}$$

<u>Remark</u>: If we define $M = 2xy^2+1$ and $N = 2x^2y$ so that our DEq^{n} is $Mdx + Ndy = 0$ we notice

$$\frac{\partial M}{\partial y} = 4xy \quad \text{and} \quad \frac{\partial N}{\partial x} = 4xy$$

This is no accident.

Def^n/ The differential form $\alpha = M\,dx + N\,dy$ is said to be exact in a rectangle R if there is a function F for which

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

So $dF = M\,dx + N\,dy$ and the eq^n $dF = M\,dx + N\,dy = 0$ is an exact eq^n as we discussed at the beginning of this section.

Digression: exact forms are closed. A Closed differential form β satisfies $d\beta = 0$. If $\beta = dF$ then the fact that $d^2 = 0$ will guarantee $d\beta = d^2F = 0 \Rightarrow \beta$ closed. Consider,

$$\begin{aligned} 0 = d(dF) &= d(M\,dx + N\,dy) \\ &= dM \wedge dx + dN \wedge dy \quad \leftarrow \boxed{\text{"}\wedge\text{" wedge product.}} \\ &= \left(\frac{\partial M}{\partial x}dx + \frac{\partial M}{\partial y}dy\right)\wedge dx + \left(\frac{\partial N}{\partial x}dx + \frac{\partial N}{\partial y}dy\right)\wedge dy \\ &= \frac{\partial M}{\partial x}\cancelto{0}{dx\wedge dx} + \frac{\partial M}{\partial y}dy\wedge dx + \frac{\partial N}{\partial x}dx\wedge dy + \frac{\partial N}{\partial y}\cancelto{0}{dy\wedge dy} \\ &= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\wedge dy \end{aligned}$$

used the fact that the wedge-product is antisymmetric.

So we find the fact $d^2 = 0$ requires that the exact differential form $M\,dx + N\,dy$ satisfies $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$ aka. $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$.

Th^m/ Suppose M, N have continous partial derivatives on R then $M\,dx + N\,dy = 0$ is an exact eq^n if and only if the <u>compatibility</u> <u>condition</u>

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

holds for all (x, y) in R.

<u>Proof</u>: See § 2.4 of text for conventional proof. Notice that my digression above also constitutes a proof if you allow me a few toys like $d^2 = 0$ and properties of the wedge product. If you're interested, in the fall I'm teaching ma 430 which will be all about these curious $\wedge$'s.

E2 Solve $(\overbrace{2xy - \sec^2 x}^{M})dx + (\overbrace{x^2 + 2y}^{N})dy = 0$.
Notice $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$ thus by Th^{m} it's exact eq^{n}.

Since it's exact $\exists\, F$ with $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = Mdx + Ndy$
We should find F subject to two partial differential eq^{n}'s,

$$\frac{\partial F}{\partial x} = 2xy - \sec^2 x \quad \& \quad \frac{\partial F}{\partial y} = x^2 + 2y$$

We can solve by integration, $\int \frac{\partial F}{\partial x}dx = F + \tilde{C}(y)$

$$F = \int (2xy - \sec^2 x)\,dx = x^2 y - \tan(x) + C_1(y)$$

Then $\frac{\partial F}{\partial y} = x^2 + \frac{\partial C_1}{\partial y} = x^2 + 2y \quad \therefore \frac{dC_1}{dy} = 2y$

notice C_1 is a function of only y.

Integrating $\frac{dC_1}{dy} = 2y$ yields $C_1 = y^2 + C_2$ hence,

$$F = x^2 y - \tan(x) + y^2 + C_2$$

Yielding sol^{n}s $\boxed{x^2 y - \tan(x) + y^2 = C}$

E3 Solve $(\underbrace{1 + e^x y + xe^x y}_{\frac{\partial F}{\partial x}})dx + (\underbrace{xe^x + 2}_{\frac{\partial F}{\partial y}})dy = 0$

Find F with:

$$F = \int \frac{\partial F}{\partial y}dy = \int (xe^x + 2)\,dy = xye^x + 2y + C_1(x)$$

$$\frac{\partial F}{\partial x} = \cancel{ye^x} + \cancel{xye^x} + \frac{dC_1}{dx} = 1 + \cancel{e^x y} + \cancel{xe^x y} \quad \therefore C_1 = x + C_2$$

Hence $F = xye^x + 2y + x + C_2$ yielding the sol^{n} $\boxed{xye^x + 2y + x = C}$

Remark: Finding F proves it is an exact eq^{n}, I could have used Th^{m} to make sure F existed before I tried to find it, but it's not necessary. If there is no such F then the eq^{n}'s $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$ will be contradictory.