Homogeneous Linear Systems with Constant Coefficients

Let A be a $(n\times n)$ constant matrix and let $x = [x_1, x_2, \dots, x_n]^T$ we wish to solve the differential eq^{n}'s

$$\boxed{x'(t) = Ax(t)} \quad \text{Eq}^{n}(1)$$

A simple guess to attempt is $x = e^{\lambda t}u$ which is inspired by our previous sucess with n^{th} order eq^{n}'s, consider

$$x' = \frac{d}{dt}\left(e^{\lambda t}u\right) = \lambda e^{\lambda t}u = \lambda x = \lambda I_n x$$

So we can rewrite Eq^{n}(1) as

$$\begin{aligned} x' = \lambda I_n x = Ax \quad &\Leftrightarrow (A-\lambda I)x = 0 \\ &\Leftrightarrow (A-\lambda I)e^{\lambda t}u = 0 \\ &\Leftrightarrow \boxed{(A-\lambda I)u = 0} \quad \text{Eq}^{n}(2) \end{aligned}$$

So our guess $x = e^{\lambda t}u$ will work only if we can find $\lambda \in \mathbb{C}$ and u a n-vector that satisfy Eq^{n}(2)

> **Def^{n}/** We say that $\lambda \in \mathbb{C}$ is an eigenvalue of a constant matrix A with eigenvector $u \neq 0$ iff
> $$Au = \lambda u \qquad \text{equivalently } (A-\lambda I)u = 0.$$

Clearly $u = 0$ is a sol^{n} to $(A-\lambda I)u = 0$, but it is <u>not</u> an eigenvector by def^{n}. Eigenvectors are nontrivial sol^{n}'s to $(A-\lambda I)u = 0$ this means that we must have $\det(A-\lambda I) = 0$ since the sol^{n} $u = 0$ is not the only sol^{n}. There will in fact be ∞ly many sol^{n}s. Anyway we find the following condition for eigenvalues λ

$$\boxed{\det(A-\lambda I) = P(\lambda) = 0} \leftarrow \underline{\text{Characteristic Eq}^{n}}$$

So important it gets a name, the polynomial P is the Characteristic Polynomial of A. See problem #41 §9.5 to gain more insight as to how this fits with old ideas.

<u>Summary:</u> $x' = Ax$ has sol^{n}'s of the form $x = e^{\lambda t} u$ if we can find λ and u such that $\det(A - \lambda I) = 0$ and $(A - \lambda I)u = 0$. We say in that case that u is an eigenvector with eigenvalue λ w.r.t. A.

E1 Find <u>eigenvalues</u> and <u>eigenvectors</u> for $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$$
\begin{aligned}
\det(A - \lambda I_2) &= \det\begin{pmatrix} 1-\lambda & 2 \\ 1 & 2-\lambda \end{pmatrix} \\
&= (1-\lambda)(2-\lambda) - 2 \\
&= 2 - 3\lambda + \lambda^2 - 2 \\
&= \lambda^2 - 3\lambda \\
&= \lambda(\lambda - 3) = 0 \Rightarrow \boxed{\lambda_1 = 0 \ \& \ \lambda_2 = 3} \text{ eigenvalues.}
\end{aligned}
$$

- We wish to find $u_1 = [u, v]^T$ with $Au_1 = \lambda_1 u_1$,

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u + 2v \\ u + 2v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Notice we get $u + 2v = 0$ twice, so let $u = s$ then $v = -u/2 = -\frac{1}{2}s$ yielding $u_1 = \begin{bmatrix} s \\ -\frac{1}{2}s \end{bmatrix} = s\begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$ for any $s \neq 0$ is an eigenvector.

- Next let $u_2 = [u, v]^T$ with $Au_2 = \lambda_2 u_2$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u + 2v \\ u + 2v \end{bmatrix} = \begin{bmatrix} 3u \\ 3v \end{bmatrix} \Rightarrow \begin{array}{l} u + 2v = 3u \\ u + 2v = 3v \end{array}$$

Both eq^{n}'s yield $u = v$ so let $u = s$ then $v = s$ for $s \neq 0$ and $u_2 = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ will be an eigenvector with e.v. 3.

<u>Remark:</u> Solving $x' = Ax$ for the $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ is easy now,

$$x(t) = c_1 e^{0t}\begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 e^{3t} \\ -\frac{1}{2}c_1 + c_2 e^{3t} \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Remark: In $\boxed{E1}$ we found 2 distinct LI sol^{n}'s to the differential eq^{n} so we could form the general sol^{n}. A natural question to ask is if there are always enough LI sol^{n}'s of the form $e^{\lambda t}u$ to form the fundamental sol^{n} set to $x' = Ax$? We should expect no, after all where could we obtain the "t" in the double root case? The analogue to the double root sol^{n} is more subtle for systems. We'll dodge the question for a while, just know that eigenvectors & eigenvalues are not enough.

$\boxed{E2}$ Find eigenvalues and eigenvectors for $A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$$0 = \begin{vmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -2-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -2-\lambda \\ 1 & 1 \end{vmatrix}$$

$$= -(\lambda+2)[(\lambda+2)^2 - 1] - [-(\lambda+2)-1] + [1+\lambda+2]$$

$$= -(\lambda+2)[\lambda^2+4\lambda+3] + 2\lambda + 6$$

$$= -(\lambda^3 + 4\lambda^2 + 3\lambda + 2\lambda^2 + 8\lambda + \cancel{6}) + 2\lambda + \cancel{6}$$

$$= -(\lambda^3 + 6\lambda^2 + 9\lambda)$$

$$= -\lambda(\lambda+3)^2 \qquad \therefore \ \underline{\lambda_1 = 0 \ \& \ \lambda_2 = -3 \text{ twice}}$$

Find $u_1 = [u\ v\ w]^T$ with $\lambda_1 = 0$

$$Au_1 = 0 \longrightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = 0 \longrightarrow \begin{array}{l} -2u + v + w = 0 \\ -(u - 2v + w = 0) \\ \hline -3u + 3v = 0 \quad \therefore \ \underline{v = u} \end{array}$$

$$\Rightarrow w = 2u - v = \underline{u = w}$$

$$\Rightarrow \boxed{u_1 = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ for } s \neq 0 \text{ is eigenvector with eigenvalue } \lambda_1 = 0}$$

$\boxed{E2}$ Continued, we now find eigenvectors with $\lambda_2 = -3$

$$(A+3I)u_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = 0 \quad \rightarrow u+v+w = 0$$

We have one eq^{n} with three unknowns $\Rightarrow$ two free parameters. Let $u = s$ and $v = r$ then $w = -u-v = -s-r$ hence,

$$u_2 = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} s \\ r \\ -s-r \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + r\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

We find two LI- eigenvectors with $\lambda_2 = -3$ namely

$$u_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \;\&\; \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \tilde{u}_2$$

This is not unique, could also use $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ those are linear combinations of u_2 and $\tilde{v}_3$. It's not hard to see all of these solve $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}u = 0$ as req^{d}.

#

Remark: We solve $x' = Ax$ for A from $\boxed{E2}$ as follows, notice even though $\lambda_2 = -3$ is repeated we do not have a te^{-3t} type term.

$$\boxed{x(t) = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 e^{-3t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}$$

Multiplicities: the eigenvalue $\lambda_2 = -3$ had multiplicity 2. We found the same # of LI eigenvectors. That is not always possible, when it's not the sol^{n} is somewhat subtle but beautiful as we shall see.

Algebraic Multiplicity = # of times repeated in $P(\lambda)$
Geometric Multiplicity = # of LI eigenvectors

In $\boxed{E2}$ the Algebraic Multiplicity = Geometric Multiplicity.

$\text{Th}^{\underline{m}}(5)$ Given an $(n\times n)$ constant matrix A with n- LI eigenvectors $u_1, u_2, \dots, u_n$ with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ respective. Then

$$\{e^{\lambda_1 t} u_1, e^{\lambda_2 t} u_2, \dots, e^{\lambda_n t} u_n\}$$

is the fundamental sol$^{\underline{n}}$ set to $x' = Ax$. Such a system has the general sol$^{\underline{n}}$

$$x(t) = c_1 e^{\lambda_1 t} u_1 + \dots + c_n e^{\lambda_n t} u_n$$

Proof:

$$W[e^{\lambda_1 t}u_1, \dots, e^{\lambda_n t}u_n] = \det\left[e^{\lambda_1 t}u_1 \middle| \cdots \middle| e^{\lambda_n t}u_n\right]$$

$$= e^{\lambda_1 t} e^{\lambda_2 t} \cdots e^{\lambda_n t} \det\left[u_1 \middle| \cdots \middle| u_n\right]$$

$$= \underbrace{e^{(\lambda_1 + \lambda_2 + \dots + \lambda_n)t}}_{\text{non zero.}} \underbrace{\det\left[u_1 \middle| \cdots \middle| u_n\right]}_{\text{non zero by LI of the eigenvectors}}$$

Clearly $e^{\lambda_j t} u_j$ is a sol$^{\underline{n}}$ for each $j = 1, 2, \dots, n$ by our previous discussions, additionally $W \neq 0 \Rightarrow$ LI of the n-sol$^{\underline{n}}$'s.

<u>Remark</u>: Some <u>eigenvalues</u> could be repeated (say k times) but in that event we've assumed that $\exists$ k - LI <u>eigenvectors</u>. That is this theorem assumes that algebraic multiplicity = geom. mult. for each eigenvalue.

E3 Given $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$ find general sol$^{\underline{n}}$ to $x' = Ax$.

$$\det[A - \lambda I] = \det\begin{bmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{bmatrix} = (2-\lambda)(-2-\lambda) + 3 = \lambda^2 - 1 = 0$$

Thus $P(\lambda) = \lambda^2 - 1 = (\lambda+1)(\lambda-1) = 0$ yielding $\underline{\lambda_1 = 1 \ \& \ \lambda_2 = -1}$.

$\underline{\lambda_1 = 1}$ $\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow \begin{matrix} 2u - 3v = u \\ u - 2v = v \end{matrix} \Rightarrow \underline{u = 3v}$

Thus we find $u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector with $\lambda_1 = 1$

$\underline{\lambda_1 = -1}$ $\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -u \\ -v \end{bmatrix} \Rightarrow \begin{matrix} 2u - 3v = -u \\ u - 2v = -v \end{matrix} \Rightarrow \underline{u = v}$

Thus we find $u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with $\lambda_2 = -1$.

General sol$^{\underline{n}}$ is thus $\boxed{x(t) = c_1 e^{t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

E4 $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

- I'll demonstrate how to interpret TI-89's results on eigen values/vectors,

$$\text{eigVl}(A) = (16.1168, -1.11684, 1.38 \times 10^{-13})$$

$$\Rightarrow \lambda_1 = 16.11, \ \lambda_2 = -1.117, \ \lambda_3 = 0$$

$$\text{eigVc}(A) = \begin{bmatrix} 0.231 & 0.786 & 0.408 \\ 0.525 & 0.086 & -0.816 \\ 0.819 & -0.612 & 0.408 \end{bmatrix}$$

$\Rightarrow$ Eigenvectors $u_1 = \begin{bmatrix} 0.231 \\ 0.525 \\ 0.819 \end{bmatrix}$ $u_2 = \begin{bmatrix} 0.786 \\ 0.086 \\ -0.612 \end{bmatrix}$ $u_3 = \begin{bmatrix} 0.408 \\ -0.816 \\ 0.408 \end{bmatrix}$

- The calculator gives messy decimal versions of the eigen vectors. It makes the length of the eigenvectors one. Its a good idea to learn how to use technology to check your work.

$Th^m(6)$ If $\lambda_1, \lambda_2, \dots \lambda_n$ are distinct eigenvalues of A then the corresponding eigenvectors $u_1, u_2, \dots, u_n$ are LI.

Proof: Let $\lambda_1 \neq \lambda_2$ with $Au_1 = \lambda_1 u_1$ and $Au_2 = \lambda_2 u_2$. Now suppose towards a contradiction that $u_1 = cu_2$ notice

$$Au_1 = Acu_2 = cAu_2$$

$$\Rightarrow \lambda_1 u_1 = c\lambda_2 u_2$$

$$\Rightarrow \lambda_1 u_1 = \lambda_2 cu_2 = \lambda_2 u_1$$

$$\Rightarrow \underbrace{(\lambda_1 - \lambda_2)}_{\text{not zero}} u_1 = 0$$

$\Rightarrow u_1 = 0$ but u_1 cannot be the zero vector since its an eigenvector $\therefore \nexists$ any c so that $u_1 = cu_2$ this means u_1 & u_2 are not linearly dependent hence they are LI.

Remark: If we have no repeated eigenvalue that gaurantees that algebraic & geometric multiplicities will be equal since for each eigenvalue we get one eigenvector. These n-eigenvectors are LI by $Th^m(6)$ so we can go back to $Th^m(5)$ and conclude

$$x(t) = c_1 e^{\lambda_1 t} u_1 + c_2 e^{\lambda_2 t} u_2 + \dots + c_n e^{\lambda_n t} u_n$$

is the general sol^n to $Ax = x'$ and by assumption

$$\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \dots \neq \lambda_n$$

Claim: If $A = A^T$ then this symmetric matrix has real eigenvalues and n-LI eigenvectors (assuming that A is $n \times n$ as usual)

Complex Eigenvalues

The examples upto now have avoided the case that $P(\lambda)$ contain an irreducible quadratic factor, that is to say we've had no complex zeroes to the characteristic eq^{n} $P(\lambda) = 0$. Lets address that case now. Consider again for $(n \times n)$ constant matrix A

$$x' = Ax$$

Look for sol^{n}'s of the form $W_1 = e^{\lambda t} u$ where $u = a + ib$ and $\lambda = \alpha + i\beta$ for a, b real entried vectors and $\alpha, \beta \in \mathbb{R}$.

$$W(t) = e^{\lambda t} u \equiv e^{(\alpha + i\beta)t}(a + ib)$$

Since we assume that $A = (a_{ij})$ is real meaning that all the entries $(a_{ij})^* = a_{ij}$ we can show that $W_2(t) = (W_1(t))^*$ is a sol^{n} with λ_1^* and u_1^*,

$$W' = AW = \lambda W \Rightarrow AW^* = \lambda^* W^*$$

This shows that W^* is eigenvector with eigenvalue λ^*. These are complex-vector-valued sol^{n}'s, we wish to find real-vector-valued sol^{n}'s. We can simply use

$$X_1 = \mathrm{Re}\{W\}$$
$$X_2 = \mathrm{Im}\{W\}$$

these will be eigenvectors associated to $\lambda = \alpha + i\beta$.

To reiterate, if $P(\lambda)$ has a complex root then the sol^{n} $e^{\lambda t}u$ needs some intrepretation if we are to find real-vector-valued sol^{n}'s. Continuing from last pg,

$$
\begin{aligned}
W(t) &= e^{(\alpha+i\beta)t}(a+ib) \\
&= e^{\alpha t}(\cos\beta t + i\sin\beta t)(a+ib) \\
&= (e^{\alpha t}\cos\beta t)a + i^2(e^{\alpha t}\sin\beta t)b \\
&\quad + i(e^{\alpha t}\cos\beta t)b + i(e^{\alpha t}\sin\beta t)a \\
&= \underbrace{e^{\alpha t}\left[(\cos\beta t)a - (\sin\beta t)b\right]}_{X_1 = Re\{W\}} + \underbrace{i e^{\alpha t}\left[(\cos\beta t)b + (\sin\beta t)a\right]}_{X_2 = Im\{W\}}
\end{aligned}
$$

Its not difficult to see that X_1 and X_2 are sol^{n}'s to to $X' = AX$ given that $W' = AW$ where $W = X_1 + iX_2$.

$$
\begin{aligned}
W' = AW \quad &\Rightarrow \quad (X_1 + iX_2)' = A(X_1 + iX_2) \\
&\Rightarrow \quad X_1' + iX_2' = AX_1 + iAX_2 \\
&\Rightarrow \quad X_1' = AX_1 \quad \text{and} \quad X_2' = AX_2 .
\end{aligned}
$$

<u>Summary</u>: If the real matrix A has complex conjugate eigenvalues $\alpha \pm i\beta$ with complex eigenvectors $a \pm ib$ then two <u>real</u>-vector-valued LI sol^{n}'s to $X' = AX$ are

$$
\begin{aligned}
X_1 &= e^{\alpha t}(\cos\beta t)a - e^{\alpha t}(\sin\beta t)b \\
X_2 &= e^{\alpha t}(\sin\beta t)a + e^{\alpha t}(\cos\beta t)b
\end{aligned}
$$

$\boxed{E5}$ $x'(t) = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} x(t)$ find general sol^{n}.

$$\det(A - \lambda I) = \det\begin{pmatrix} -1-\lambda & 2 \\ -1 & -3-\lambda \end{pmatrix} = (\lambda+1)(\lambda+3)+2 = \lambda^2 + 4\lambda + 5$$

So $\det(A - \lambda I) = 0 \Rightarrow \lambda = \dfrac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i = \alpha \pm i\beta$.

<u>Lets find the eigenvector of $-2+i$ w.r.t. $A = \begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}$</u>

$$\begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = (-2+i)\begin{bmatrix} u \\ v \end{bmatrix}$$

$$-u + 2v = (-2+i)u \Rightarrow 2v = (-1+i)u$$

let $u = 2$ then $v = -1+i$ and we find eigenvector

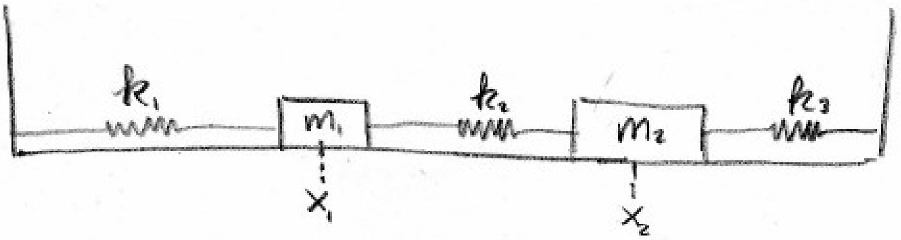
$$u_1 = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2 \\ -1+i \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{a} + i\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b} = a + ib$$

Thus we find general sol^{n},

$$\boxed{\begin{aligned} x(t) = {} & c_1 e^{-2t}\left[\cos t \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] + \\ & + c_2 e^{-2t}\left[\sin t \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] \end{aligned}}$$

<u>Remark</u>: Although we have two eigenvalues $\alpha + i\beta$ and $\alpha - i\beta$ it is sufficient to find the eigenvector $a+ib$ for $\alpha - i\beta$ because the eigenvector for $\alpha - i\beta$ is simply $a - ib$. In principle you might ask why find $a+ib$ instead of $a-ib$? In fact it does not matter, either choice gives same general sol^{n} (it would change meaning of c_1 & c_2 but those are arbitrairy anyway.)

An example of complex eigenvalues physically



Newton's 2nd law yields a DEq^n for each mass

m_1 $\quad m_1 x_1'' = -k_1 x_1 + k_2 (x_2 - x_1)$

m_2 $\quad m_2 x_2'' = -k_2 (x_2 - x_1) - k_3 x_2$

We can put this in standard form by introducing $y_1 \equiv x_1$, $y_2 = x_1'$, $y_3 = x_2$, $y_4 = x_2'$ with $y(t) = [y_1, y_2, y_3, y_4]^T$ then

$$
\begin{aligned}
y_1' &= x_1' = y_2 \\
y_2' &= x_1'' = (-k_1/m_1) x_1 + (k_2/m_1)(x_2 - x_1) \\
y_3' &= x_2' = y_4 \\
y_4' &= x_2'' = -k_2/m_2 (x_2 - x_1) - (k_3/m_2) x_2
\end{aligned}
$$

$$
\boxed{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-1}{m_1}(k_1 + k_2) & 0 & k_2/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k_2/m & 0 & -\frac{1}{m_2}(k_2 + k_3) & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}}
$$

As you might expect looking ahead physically for oscillatory motion, this system will have imaginary eigenvalues $\pm i\beta_1$ and $\pm i\beta_2$ these are the <u>normal</u> or <u>natural</u> frequencies. When external forces are put on such a system one wants to avoid oscillatory forces of the same natural frequencies, otherwise the system will suffer resonance, and unbounded build-up of energy.