Ma 341, selected homework sol^{n}'s

§1.1 #1 $\quad 3\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 9x = 2\cos(3t)$ $\quad$ dependent variable $= x$, independent variable $= t$

is a nonhomogeneous, 2nd order, ODE, which is linear.
This equation describes motion of a spring moving in a media and the mass has an external force $\propto$ to $\cos(3t)$ applied.
Or this could describe an RLC circuit with an oscillating source voltage of $2\cos(3t)$. The possibilities are endless really.

§1.1 #3 $\quad \frac{dy}{dx} = \frac{y(2-3x)}{x(1-3y)}$: 1st order nonlinear ODE, dependent var. $= y$, indep. var. $= x$

§1.1 #9 $\quad x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$: 2nd order linear ODE, dep. var. $= y$, indep. var. $= x$.

§1.1 #11 $\quad \frac{\partial N}{\partial t} = \frac{\partial^2 N}{\partial r^2} + \frac{1}{r}\frac{\partial N}{\partial r} + kN$: 2nd order PDE

§1.2 #1 (a.) $\phi(x) = 2x^3$ has $\phi'(x) = 6x^2$ thus

$$x\cdot\phi'(x) = x\,6x^2 = 6x^3 = 3(2x^3) = 3\phi(x)$$

$\therefore\ \phi(x) = 2x^3$ is a sol^n to $x\frac{dy}{dx} = 3y$.

(b.) Show $\phi(x) = e^x - x$ solves $y' + y^2 = e^{2x} + (1-2x)e^x + x^2 - 1$.

Let $y = \phi(x)$.

$$y' = e^x - 1$$

$$y^2 = (e^x - x)^2 = (e^x)^2 - 2xe^x + x^2$$

But $(e^x)^2 = e^{2x}$ thus

$$y'' + y^2 = e^x - 1 + e^{2x} - 2xe^x + x^2$$

$$= e^{2x} + (1-2x)e^x + x^2 - 1 \quad \therefore\ \phi(x) \text{ is a sol}^n.$$

§1.2 #3 $\quad \boxed{y'' + y = x^2 + 2}$ (*) is $y = \sin(x) + x^2$ a sol^n of (*)?

Well $y'' = \frac{d}{dx}\frac{d}{dx}(\sin(x) + x^2) = \frac{d}{dx}(\cos(x) + 2x) = -\sin(x) + 2$

Thus $y'' + y = -\cancel{\sin(x)} + 2 + \cancel{\sin(x)} + x^2 = x^2 + 2$ thus

$y = \sin(x) + x^2$ is a sol^n to (*).

§1.2 #5 $\frac{dx}{dt} + tx = \sin(2t)$ is $x = \cos(2t)$ a sol^{n} ?

Let $x = \cos(2t)$ then,

$$\frac{dx}{dt} + tx = -2\sin(2t) + t\cos(2t) \stackrel{(?)}{=} \sin(2t)$$ answer to (?) is No.

Equating coefficients of linearly dependent functions,
{more on this later}

$\sin(2t): \quad -2 = 1$

$t\cos(2t): \quad 1 = 0$

The idea here I'm using is that $\sin(t)$ and $\cos(t)$ are never both zero for same t, thus they are independent in the sense that if $A\sin(t) + B\cos(t) = C\sin(t) + D\cos(t) \Rightarrow$ $A = C$ $(\sin(t))$, $B = D$ $(\cos(t))$

§1.2 #9 $x^2 + y^2 = 6 \Rightarrow 2x + 2y\frac{dy}{dx} = 0$ (just differentiate w.r.t. x)

$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$ thus this is not an implicit sol^{n} to $\frac{dy}{dx} = \frac{x}{y}$

(in fact $x^2 + y^2 = 6$ gives an implicit sol^{n} to $\frac{dy}{dx} = -\frac{x}{y}$.)

§1.2 #11 $e^{xy} + y = x - 1$ diff. w.r.t. x; assume $y = y(x)$,

$$e^{xy}\frac{d}{dx}(xy) + \frac{dy}{dx} = 1$$

product rule

$$e^{xy}\left(y + x\frac{dy}{dx}\right) + \frac{dy}{dx} = 1$$

$$\frac{dy}{dx}\left(xe^{xy} + 1\right) = 1 - ye^{xy} \Rightarrow \frac{dy}{dx} = \frac{1 - ye^{xy}}{xe^{xy} + 1}$$

Multiplying by e^{-xy} on top & bottom yields

$$\frac{dy}{dx} = \frac{e^{-xy} - y}{x + e^{-xy}}$$

thus $e^{xy} + y = x - 1$ implicitly solves $\frac{dy}{dx} = \frac{e^{-xy} - y}{e^{-xy} + x}$.

§1.2 #21 Let $\phi(x) = x^m$, what must m be to solve $3x^2y'' + 11xy' - 3y = 0$

Let $y = \phi(x)$ then, $\phi'(x) = mx^{m-1}$ & $\phi''(x) = m(m-1)x^{m-2}$

$$\begin{aligned} 3x^2y'' + 11xy' - 3y &= 3x^2 m(m-1)x^{m-2} + 11x\,mx^{m-1} - 3x^m \\ &= (3m(m-1) + 11m - 3)\,x^m \quad \text{(cool.)} \\ &= 0 \end{aligned}$$

Thus we need $3m(m-1) + 11m - 3 = 0$, a.k.a. $3m^2 + 8m - 3 = 0$

Which has well-known sol^{n} $m = \frac{-8 \pm \sqrt{64+36}}{6} = \boxed{\frac{-18}{6} \text{ or } \frac{2}{6} = m}$

§1.2 # 23 $\frac{dY}{dx} = x^3 - y^3$ and $Y(0) = 6$ does Th$^{\underline{m}}$ (1) tell us that this IVP has a unique sol$^{\underline{n}}$?

Th^{m}(1) says if $\frac{dY}{dx} = f(x,Y)$ and $\frac{\partial f}{\partial Y}$ and f are continuous in some rectangle R containing the intial condition (x_o, Y_o) then $\exists$ some interval containing x_o such that the DE$q^{\underline{n}}$ will have a unique sol$^{\underline{n}}$.

For our problem here $f(x,y) = x^3 - y^3$ and $\frac{\partial f}{\partial y} = 3y^2$ these are both polynomial fncts. of x & y (polynomial functions are continous everywhere since $\lim_{\vec{x} \to \vec{a}} P(x_1, x_2, \dots, x_n) = P(a_1, a_2, \dots, a_n)$)

Thus by Th^{m} (1) we get a unique sol$^{\underline{n}}$ for some nbhd of zero.

§1.2 # 30 Consider $e^{xY} + x + Y = 0$. We wish to show this eq$^{\underline{n}}$ implicitly defines Y as a fnct. of x near $(0, -1)$. Use the implicit fnct. Th^{m},

$$G(x, Y) = e^{xY} + x + Y$$

Note $G(0,-1) = e^0 + 0 - 1 = 1 - 1 = 0$. Also calculate

$$\frac{\partial G}{\partial x} = Ye^{xY} + 1 \quad \text{and} \quad \frac{\partial G}{\partial Y} = xe^{xY} + 1$$

these partials are continous everywhere. Then note

$$\left.\frac{\partial G}{\partial Y}\right|_{(0,-1)} = 0 \cdot e^0 + 1 = 1 \neq 0$$

$\therefore$ by implicit fnct. Th$^{\underline{m}}$ $\exists\, \phi(x)$ such that $G(x, \phi(x)) = 0$. Hence $Y = \phi(x)$ near $(0,-1)$.

Remark: the Th$^{\underline{m}}$ tells us there exists such a function ϕ, it is however silent on how to calculate ϕ's explicit formula, in fact that may be impossible in closed form. Existence is never the less important since we can justifiably find approximate sol$^{\underline{n}}$'s by replacing all the eq$^{\underline{n}}$'s expressions with power series centered near the point of interest... (we won't do this, just mentioning it.)

§2.2#1 $\frac{dy}{dx} = 2y^3 + y + 4 \Rightarrow \frac{dy}{2y^3+y+4} = dx$ (seperated variables!) So it's seperable.

§2.2#2 $\frac{dy}{dx} = \sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$ (trig-identity we'll derive later in lecture.)

$= \cos(x)\,[\underbrace{\tan(x)\cos(y) + \sin(y)}]$

no way to factor out $\tan(x)$ w/o adding x-dependence to $\sin(y)$ term. $\Rightarrow$ Cannot factor into $g(x)P(y)$.

§2.2#3 $\frac{dy}{dx} = \frac{ye^{x+y}}{x^2+2} = ye^y\left(\frac{e^x}{x^2+2}\right) = P(y)\,g(x)$ with the obvious choices.

§2.2#4 $\frac{ds}{dt} = t\ln(s^{2t}) + 8t^2$

$= t \cdot 2t\ln(s) + 8t^2$

$= t^2(2\ln(s) + 8) \Rightarrow \frac{ds}{2\ln(s)+8} = t^2 dt$, seperable.

§2.2#7 $\frac{dy}{dx} = \frac{1-x^2}{y^2} \Rightarrow y^2 dy = (1-x^2)dx$

integrating both sides. $\Rightarrow \boxed{y^3/3 = x - x^3/3 + C}$ (implicit sol^n for DEq^n)

§2.2#8 $\frac{dy}{dx} = \frac{1}{xy^3} \Rightarrow \int y^3 dy = \int \frac{dx}{x} \Rightarrow \boxed{\frac{1}{4}y^4 = \ln|x| + C}$

§2.2#9 $\frac{dy}{dx} = y(2+\sin(x)) \Rightarrow \int \frac{dy}{y} = \int (2+\sin(x))dx \Rightarrow \boxed{\ln|y| = 2x - \cos(x) + C}$

§2.2#10 $\frac{dx}{dt} = 3xt^2 \Rightarrow \int \frac{dx}{x} = \int 3t^2 dt \Rightarrow \boxed{\ln|x| = t^3 + C}$

§2.2#11 $\frac{dy}{dx} = \frac{\sec^2(y)}{1+x^2} \Rightarrow \int \frac{dy}{\sec^2(y)} = \int \frac{dx}{1+x^2}$ note $\frac{1}{\sec^2 y} = \cos^2(y)$

$\int \frac{dy}{\sec^2 y} = \int \cos^2 y\, dy \overset{(*)}{=} \int \frac{1}{2}(1+\cos(2y))dy = \frac{1}{2}(y + \frac{1}{2}\sin(2y)) + C$

$\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + C$ (*) trig identity

Thus we find $\boxed{\frac{1}{2}y + \frac{1}{4}\sin(2y) = \tan^{-1}(x) + C}$

§2.2 #12 $x\frac{dv}{dx} = \frac{1-4v^2}{3v} \Rightarrow \frac{3v\,dv}{1-4v^2} = \frac{dx}{x}$ then consider,

$$\int \frac{3v\,dv}{1-4v^2} = \int \frac{3\frac{-1}{8}du}{u} = -\frac{3}{8}\ln|u| + c = -\frac{3}{8}\ln|1-4v^2| + c$$

$u = 1-4v^2$
$du = -8v\,dv$

thus $\boxed{-\frac{3}{8}\ln|1-4v^2| = \ln|x| + c}$

Remark: it would be a good exercise to find explicit sol^{n}'s for # 8, 9, 10 & 12. That is do some algebra, solve each implicit sol^{n} for the dependent variable (y, y, x, v respectively)

§2.2 #20 $\frac{dy}{dx} = \frac{3x^2+4x+2}{2y+1} \Rightarrow (2y+1)dy = (3x^2+4x+2)dx$

$\Rightarrow y^2 + y = x^3 + 2x^2 + 2x + C$

$y(0) = -1 \Rightarrow (-1)^2 - 1 = 0 + 2(0)^2 + 2(0) + C \Rightarrow \boxed{C = 0}$

therefore the sol^{n} with $y(0) = -1$ is $\boxed{y^2 + y = x^3 + 2x^2 + x}$

§2.2 #21 $\frac{dy}{dx} = 2\sqrt{y+1}\cos(x) \Rightarrow \int \frac{dy}{\sqrt{y+1}} = \int 2\cos(x)\,dx$

$$\int \frac{dy}{\sqrt{y+1}} = \int \frac{du}{\sqrt{u}} = \frac{u^{\frac{1}{2}}}{1/2} + C = 2\sqrt{y+1} + C$$

letting $u = y+1$
so $du = dy$

$\therefore 2\sqrt{y+1} + C = 2\sin(x)$

We know $y(\pi) = 0 \Rightarrow 2\sqrt{0+1} + C = 2\sin(\pi)$

$\Rightarrow 2 + C = 0 \Rightarrow C = -2$

$\Rightarrow \boxed{2\sqrt{y+1} - 2 = 2\sin(x)}$

§2.2 #22 $x^2dx + 2y\,dy = 0 \Rightarrow \int x^2dx = \int -2y\,dy$

thus $\frac{1}{3}x^3 = -y^2 + C$. We know $y(0) = 2 \Rightarrow 0 = -4 + C \therefore \underline{C = 4}$

Hence $\boxed{\frac{1}{3}x^3 = y^2 + 4}$.

§2.2 #23 $\frac{dy}{dx} = 2x\cos^2(y) \Rightarrow \int \frac{dy}{\cos^2 y} = \int 2x\,dx \overset{(*)}{\Rightarrow} \tan(y) = x^2 + C.$

(*) $\left(\frac{1}{\cos^2 y} = \sec^2 y \text{ and } \int \sec^2 y\,dy = \tan(y) + c\right)$ then $y(0) = \frac{\pi}{4} \Rightarrow \underline{\tan(\pi/4) = 0 + C}$

Now $\tan(\pi/4) = 1$ thus $\boxed{\tan(y) = x^2 + 1}$

§2.2 #24 $\frac{dY}{dx} = 8x^3 e^{-2Y} \Rightarrow e^{2Y}dY = 8x^3 dx$
$\Rightarrow \frac{1}{2}e^{2Y} = 2x^4 + C$

$y(1) = 0 \Rightarrow \frac{1}{2}e^{(0)} = 2(1)^4 + C \Rightarrow \frac{1}{2} = 2 + C \Rightarrow \underline{C = -3/2}$

thus $\boxed{\frac{1}{2}e^{2Y} = 2x^4 - \frac{3}{2}}$ ok I'll find the explicit sol$^{\underline{n}}$ for this one, $e^{2Y} = 4x^4 - 3 \Rightarrow \underline{2Y = \ln(4x^4 - 3)}$
$\Rightarrow \boxed{Y = \ln\left(\sqrt{4x^4 - 3}\right)}$.

§2.2 #25 $\frac{dy}{dx} = x^2(1+y) \Rightarrow \int \frac{dy}{1+y} = \int x^2 dx \Rightarrow \ln|1+y| = \frac{x^3}{3} + C.$

Now $y(0) = 3 \Rightarrow \ln|4| = 0^3/3 + C \therefore C = \ln(4)$ hence

$\boxed{\ln|1+y| = \frac{1}{3}x^3 + \ln(4)}$

§2.2 #26 $\sqrt{y}\,dx + (1+x)dy = 0 \Rightarrow \int \frac{dy}{\sqrt{y}} = \int \frac{-dx}{1+x}$

thus integrating yields $2\sqrt{y} = -\ln|1+x| + C$

Then $y(0) = 1 \Rightarrow 2 = -\ln(1) + C = C \therefore \boxed{2\sqrt{y} = -\ln|1+x| + 2}$

Or if you want $\boxed{y = \left[\ln\left(\frac{1}{\sqrt{|1+x|}}\right) + 2\right]^2}$

§2.2 #29 Consider $\frac{dY}{dx} = \sqrt[3]{Y}$

a.) $\int \frac{dY}{\sqrt[3]{Y}} = \int dx \Rightarrow \frac{3}{2}Y^{2/3} = x + \tilde{C} \Rightarrow \boxed{Y = \left(\frac{2}{3}x + C\right)^{3/2}}$

b.) Consider $Y = \left(\frac{2x}{3}\right)^{3/2}$ notice $\frac{dY}{dx} = \frac{3}{2}\left(\frac{2x}{3}\right)^{1/2}\left(\frac{2}{3}\right) = \left(\frac{2x}{3}\right)^{1/3} = \sqrt[3]{Y}$

and $Y(0) = \left(\frac{2(0)}{3}\right)^{3/2} = 0$ thus $Y = \left(\frac{2x}{3}\right)^{3/2}$ is a sol$^{\underline{n}}$ with $y(0) = 0$.

c.) Notice $y = 0$ is also a sol$^{\underline{n}}$ with $y(0) = 0$. Clearly the constant fnct. has $\frac{d}{dx}(0) = 0$ thus $\frac{dY}{dx} = 0 = \sqrt[3]{0} = 0$.

Thus : Sol$^{\underline{n}}$'s to arbitrairy IVP are <u>not</u> unique.

d.) Here $\frac{dY}{dx} = f(x,Y)$ has $f(x,Y) = \sqrt[3]{Y}$ notice

$\frac{\partial f}{\partial y} = \frac{1}{3}\frac{1}{(\sqrt[3]{y})^2} \leftarrow$ <u>not</u> continous at $(0,0)$, thus the criteria of the unique sol$^{\underline{n}}$ to IVP Th$^{\underline{m}}$ on pg. 12 are not met.

§2.2 #31

a.) $\frac{dy}{dx} = xy^3 \Rightarrow \int \frac{1}{y^3}\,dy = \int x\,dx \Rightarrow \boxed{\frac{-1}{y^2} = x^2 + C}$

b.) $y(0) = 1 \Rightarrow -1 = 0 + C \quad \therefore \boxed{-1/y^2 = x^2 - 1}$ I

$y(0) = 1/2 \Rightarrow -4 = C \quad \therefore \boxed{-1/y^2 = x^2 - 4}$ II

$y(0) = 2 \Rightarrow -1/4 = C \quad \therefore \boxed{-1/y^2 = x^2 - 1/4}$ III

oops. these are implicit see text for explicit versions

c.) I $\frac{-1}{y^2} < 0 \Rightarrow x^2 - 1 < 0 \Rightarrow x^2 < 1 \Rightarrow -1 < x < 1.$

II $\frac{-1}{y^2} < 0 \Rightarrow x^2 - 4 < 0 \Rightarrow -2 < x < 2.$

III $\frac{-1}{y^2} < 0 \Rightarrow x^2 - 1/4 < 0 \Rightarrow \frac{-1}{2} < x < \frac{1}{2}.$

d.) $y(0) = a > 0 \Rightarrow \frac{-1}{a^2} = C \quad \therefore \frac{-1}{y^2} = x^2 - \frac{1}{a^2} \Rightarrow y = \frac{1}{\sqrt{1/a^2 - x^2}}$

domain of sol^n given by $\frac{1}{a^2} - x^2 > 0 \Rightarrow \frac{1}{a^2} > x^2 \Rightarrow \frac{-1}{a} < x < \frac{1}{a}$

as $a \to 0$ (from right where $a > 0$) then $\frac{\pm 1}{a} \to \pm\infty$
hence domain $\to (-\infty, \infty)$. Alternatively
as $a \to \infty$ we have $\pm\frac{1}{a} \to 0$ hence domain is $\{0\}$.

§2.3 #1 $x^2 \frac{dy}{dx} + \cos(x) = y$, linear but not seperable.

§2.3 #2 $\frac{dx}{dt} + xt = e^x$, neither.

§2.3 #3 $x\frac{dx}{dt} + t^2 x = \sin(t)$, neither ($x\frac{dx}{dt}$ non linear term)

§2.3 #6 $3r = \frac{dr}{d\theta} - \theta^3$, linear, not seperable.

<u>Remark</u>: to prove an eq^n is seperable we merely have to factor the expression appropriately. When it is not seperable it's difficult to prove it cannot be seperated, basically we just have to see it.

§2.3 #7 (long version) $\dfrac{dy}{dx} - y = e^{3x}$ is in standard form with $P(x) = -1$ and $Q(x) = e^{3x}$

Calculate $\mu(x) = \exp\left(\int -1\,dx\right) = \exp(-x)$. Then multiply the DEq.ⁿ by $\mu(x)$,

$$\underbrace{e^{-x}\frac{dy}{dx} - e^{-x}y}_{\dfrac{d}{dx}\left(e^{-x}y\right)} = e^{-x}e^{3x} = e^{2x}$$

$$\frac{d}{dx}\left(e^{-x}y\right) = e^{2x}$$

product rule in reverse.

Integrate → use FTC $\underbrace{e^{-x}y = \frac{1}{2}e^{2x} + c}_{\text{☆}} \Rightarrow \boxed{y = \frac{1}{2}e^{3x} + ce^{x}}$

(dividing by e^{-x})

§2.3 #7 (Short version) $P(x) = -1$ and $Q(x) = e^{3x}$

$$\begin{aligned} y &= \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)\,dx + c\right] \quad : \text{Eq.ⁿ (8) of pg. (50).} \\ &= \frac{1}{e^{-x}}\left[\int e^{-x}e^{3x}\,dx + c\right] \\ &= \frac{1}{e^{-x}}\left[\frac{1}{2}e^{2x} + c\right] = \frac{1}{2}e^{3x} + ce^{x} \end{aligned}$$

Remark: I expect you to be able to show <u>why</u> this (long version) method works. It's name is the "<u>integrating factor method</u>". Here $\mu(x)$ is the integrating factor.

<u>Remark</u>: Notice that we need not add a constant upon integrating $P(x)$ in $\mu(x) = \exp\left(\int P(x)\,dx\right)$. If we did it would cancel when we divide by $\mu(x)$ to solve for y. On the other hand in the final integration ☆ we have no such expectation for that C to be cancelled. It is thus our custom to omit the integration constant when calculating $\mu(x)$.

§2.3 #8

$$\frac{dy}{dx} = \frac{y}{x} + 2x + 1 \Rightarrow \overbrace{\frac{dy}{dx} - \frac{1}{x}y = 2x+1}^{\text{standard form.}}$$

$$\mu(x) = \exp\left(\int \frac{-1}{x}dx\right) = \exp(-\ln|x|) = \exp\left(\ln\left|\frac{1}{x}\right|\right) = \frac{1}{|x|}$$

Multiply by $\mu(x)$, assume $x > 0 \Rightarrow |x| = x$.

$$\underbrace{\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y}_{} = \frac{1}{x}(2x+1) = 2 + \frac{1}{x}$$

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = 2 + \frac{1}{x}$$

integrating both sides, use for LHS that $\int \frac{d}{dx}(f(x))dx = f(x)$.

$$\frac{1}{x}y = \int\left(2 + \frac{1}{x}\right)dx = 2x + \ln|x| + C$$

$$\boxed{y = 2x^2 + x\ln(x) + cx} \quad (x > 0)$$

Sol$^{\text{n}}$ for $x < 0$ similar except $|x| = -x$.

<u>Remark</u>: You have two-options for the test.

1.) go through the steps each time
2.) <u>prove</u> f-la (8) in general then apply result.

You cannot just plug & chug from (8). You must demonstrate you understand why it works by 1) or 2.)

§2.3 #9 $\frac{dr}{d\theta} + (\tan\theta)r = \sec\theta$ $\quad \mu(\theta) = \exp\left(\int \tan\theta\, d\theta\right) = \exp(\ln|\sec\theta|)$

$$|\sec\theta|\frac{dr}{d\theta} + |\sec\theta|\tan\theta\, r = |\sec\theta|\sec\theta$$

$$\underbrace{\sec\theta\frac{dr}{d\theta} + \sec\theta\tan\theta\, r}_{} = \sec^2\theta$$

$$\frac{d}{d\theta}(r\sec\theta) = \sec^2\theta \Rightarrow r\sec\theta = \int \sec^2\theta\, d\theta = \tan\theta + C$$

$$\Rightarrow r = \frac{\tan\theta}{\sec\theta} + \frac{C}{\sec\theta}$$

$$\Rightarrow \boxed{r = \sin\theta + (\cos\theta)C}$$

§2.3 #10 $$Y + \frac{X}{2}\frac{dY}{dX} = \frac{1}{2X^3}$$

$$\frac{dY}{dX} + \frac{2}{X}Y = \frac{1}{X^4} \Rightarrow \mu(x) = \exp\left(\int \frac{2}{x}dx\right) = e^{2\ln|x|} = |x|^2 = x^2.$$

$$\underbrace{X^2\frac{dY}{dX} + 2XY}_{} = \frac{X^2}{X^4}$$

$$\frac{d}{dx}(X^2Y) = \frac{1}{X^2} \Rightarrow X^2Y = \int\frac{dx}{X^2} = \frac{-1}{X} + C \therefore \boxed{Y = \frac{-1}{X^3} + \frac{C}{X^2}}$$

§2.3 #11 $$(t+y+1)dt - dy = 0 \Rightarrow \frac{dy}{dt} = t+y+1 \Rightarrow \underbrace{\frac{dy}{dt} - y = t+1}_{\text{(standard form)}}$$

$$\mu = \exp\left(\int -dt\right) = \exp(-t)$$

$$\underbrace{e^{-t}\frac{dy}{dt} - e^{-t}y}_{} = e^{-t}(t+1)$$

$$\int \frac{d}{dt}(e^{-t}y) = \int(te^{-t} + e^{-t})dt$$

$$= \int \underbrace{t}_{u}\underbrace{e^{-t}dt}_{dv} + \int e^{-t}dt$$

$$= -te^{-t} + \int e^{-t}dt - e^{-t}$$

$$= -te^{-t} - e^{-t} - e^{-t} + C \Rightarrow e^{-t}y = -e^{-t}(2+t) + C$$

$$\Rightarrow \boxed{Y = -2 - t + Ce^{t}}$$

§2.3 #12 $$\frac{dy}{dx} + 4y = x^2e^{-4x} \Rightarrow \mu(x) = \exp\left(\int 4dx\right) = e^{4x}.$$

$$e^{4x}\frac{dy}{dx} + 4e^{4x}y = x^2 \Rightarrow \frac{d}{dx}(e^{4x}y) = x^2$$

$$\Rightarrow e^{4x}y = \int x^2dx = \frac{x^3}{3} + C$$

$$\Rightarrow \boxed{Y = e^{-4x}\left(\tfrac{1}{3}x^3 + C\right)}$$

§2.3 #18 $$\frac{dy}{dx} + 4y = e^{-x} \Rightarrow \mu = \exp\left(\int 4dx\right) = e^{4x}$$

$$e^{4x}\frac{dy}{dx} + 4e^{4x}y = e^{4x}e^{-x} \Rightarrow \frac{d}{dx}(e^{4x}y) = e^{3x} \Rightarrow e^{4x}y = \frac{1}{3}e^{3x} + C$$

$$y(0) = 4/3 \Rightarrow e^0\frac{4}{3} = \frac{1}{3}e^0 + C \Rightarrow \underline{C = 1}$$

$$\boxed{Y = \frac{1}{3}e^{-x} + e^{-4x}}$$

(#17 → #21 same as #7 → #16 just have to figure out what the constant should be)

§2.3 #22 $\sin(x)\frac{dy}{dx} + y\cos(x) = x\sin(x)$

$$\frac{dy}{dx} + \frac{\cos(x)}{\sin(x)}y = x$$

$$\mu = \exp\left(\int \frac{\cos(x)dx}{\sin(x)}\right) = \exp\left(\int \frac{du}{u}\right) = \exp(\ln|\sin(x)|) = |\sin(x)|$$

$u = \sin(x)$
$du = \cos(x)dx$

$$|\sin(x)|\frac{dy}{dx} + \frac{|\sin(x)|}{\sin(x)}\cos(x)y = |\sin(x)|x$$

$$\sin(x)\frac{dy}{dx} + \cos(x)y = x$$

$$\frac{d}{dx}(\sin(x)y) = x \Rightarrow y\sin(x) = \frac{x^2}{2} + C$$

$$y(\pi/2) = 2 \Rightarrow 2\sin(\pi/2) = \frac{\pi^2}{8} + C \quad \therefore \quad \underline{C = 2 - \pi^2/8}$$

Thus $\boxed{y = \frac{1}{\sin(x)}\left(\frac{x^2}{2} + 2 - \frac{\pi^2}{8}\right)}$

§2.3 #29 $\frac{dy}{dx} = \frac{1}{e^{4y}+2x} \longrightarrow \frac{dx}{dy} = e^{4y} + 2x$

Then $\frac{dx}{dy} - 2x = e^{4y}$ thus $\mu(y) = \exp\left(\int -2dy\right) = e^{-2y}$.

$$\underbrace{e^{-2y}\frac{dx}{dy} - 2e^{-2y}x}_{} = e^{2y}$$

$$\frac{d}{dy}(e^{-2y}x) = e^{2y} \rightarrow e^{-2y}x = \frac{1}{2}e^{2y} + C$$

$$\rightarrow \boxed{x = \frac{1}{2}e^{4y} + Ce^{2y}}$$

§2.4 #1 $\underbrace{(x^2y + x^4\cos x)}_{M}dx - \underbrace{x^3}_{N}dy = 0$

$\frac{\partial M}{\partial y} = x^2$ $\quad \frac{\partial N}{\partial x} = 3x^2$ $\quad \therefore$ not exact (by Thm (2) p. 61.)

However $\frac{dy}{dx} = \frac{x^2}{x^3}y + \frac{x^4}{x^3}\cos(x)$ so it's linear, but not seperable.

§2.4 #4 $\underbrace{\sqrt{-2y-y^2}}_{M}dx + \underbrace{(3+2x-x^2)}_{N}dy = 0$

$\frac{\partial M}{\partial y} = \frac{-2-2y}{2\sqrt{-2y-y^2}}$ $\quad \frac{\partial N}{\partial x} = 2-2x$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \therefore$ not exact.

§2.4#8 $\underbrace{[2x + y\cos(xy)]}_{M}dx + \underbrace{[x\cos(xy) - 2y]}_{N}dy = 0$

$\frac{\partial M}{\partial y} = \cos(xy) - y\sin(xy)\cdot x \qquad \frac{\partial N}{\partial x} = \cos(xy) - x\sin(xy)\cdot y$

Thus $\frac{\partial M}{\partial y} = \frac{\partial M}{\partial x}$ thus this eq^{n} is exact. That means $\exists\, F(x,y)$ such that $dF = Mdx + Ndy$. What's F?

§2.4#13 $\underbrace{(t/y)}_{M}dy + \underbrace{(1+\ln(y))}_{N}dt = 0 \qquad \frac{\partial M}{\partial t} = \frac{1}{y} = \frac{\partial N}{\partial y}$ exact ✓

So there exists $F(y,t)$ with $dF = Mdy + Ndt$ $\left(M = \frac{\partial F}{\partial y} \quad N = \frac{\partial F}{\partial t}\right)$

$$\frac{\partial F}{\partial y} = M \longrightarrow F(y,t) = \int (t/y)\,dy = t\ln(y) + C_1(t)$$

$$\frac{\partial F}{\partial t} = N \rightarrow F(y,t) = \int (1+\ln(y))dt = t\ln(y) + t + C_2(y)$$

Comparing, we deduce that $\boxed{F(y,t) = t\ln(y) + t + C}$ then the sol^{n} to the differential eq^{n} is $F(y,t) = \boxed{0 = t\ln(y) + t + C}$ (sol^{n}.) This is a sol^{n} because dF is the DEq^{n} we began with:

$$dF = \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial t}dt = (t/y)\,dy + (\ln(y)+1)\,dt = d(0) = 0.$$

the circle is complete.

§2.4#14 $e^{t}(y-t)dt + (1+e^{t})dy = Mdt + Ndy = 0$, $\left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}\right)$ exact. ✓

$$F(y,t) = \int \frac{\partial F}{\partial y}dy = \int (1+e^{t})dy = y(1+e^{t}) + C_1(t)$$

$$\frac{\partial F}{\partial t} = \cancel{y}e^{t} + \frac{\partial C_1}{\partial t} = e^{t}(\cancel{y}-t) \Rightarrow \frac{\partial C_1}{\partial t} = -te^{t}$$

Then $\int te^{t}dt = te^{t} - e^{t} + C$, thus as $C_1 = C_1(t)$ we have $\frac{\partial C_1}{\partial y} = 0$, $C_1(t) = \int \frac{\partial C_1}{\partial t}dt = -\int te^{t}dt = e^{t} - te^{t} + C$. Again $dF = 0$ is our differential equation so $F = 0$ gives sol^{n}, that is

$$\boxed{y(1+e^{t}) + e^{t} - te^{t} + C = 0}$$

§2.4 #15 $\frac{\partial}{\partial\theta}(\cos\theta) = -\sin\theta = \frac{\partial}{\partial r}(r\sin\theta - e^{\theta})$ so it's exact, find F as usual,

$$\underbrace{\cos\theta}_{\frac{\partial F}{\partial r}}\, dr - \underbrace{(r\sin\theta - e^{\theta})}_{\frac{\partial F}{\partial\theta}}\, d\theta = 0 = dF \quad \text{now find } F$$

$$F(r,\theta) = \int \frac{\partial F}{\partial r}\, dr = \int \cos\theta\, dr = r\cos\theta + C_1(\theta)$$

$$\frac{\partial F}{\partial\theta} = \cancel{-r\sin\theta} + \frac{\partial C_1}{\partial\theta} = \cancel{-r\sin\theta} + e^{\theta} \Rightarrow \frac{dC_1}{d\theta} = e^{\theta} \Rightarrow C_1 = e^{\theta} + C$$

Thus the sol^{n} is $F = \boxed{r\cos\theta + e^{\theta} + C = 0}$ implicitly defines sol^{n}.

§2.4 #16 You can verify it's exact. Find F such that

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = (ye^{xy} - 1/y)dx + (xe^{xy} + x/y^2)dy$$

Eyeballing it you can read off:

$$F = e^{xy} - x/y + C$$

Then sol^{n} is $\boxed{e^{xy} - x/y + C = 0}$

§2.4 #17 $\frac{1}{y}dx - (3y - x/y^2)\,dy$ note $\frac{\partial}{\partial y}\left(\frac{1}{y}\right) = \frac{-1}{y^2}$ but $\frac{\partial}{\partial x}\left(-3y + \frac{x}{y^2}\right) = \frac{1}{y^2}$ $\boxed{\text{not exact}}$

§2.4 #18

$$\left.\begin{aligned} 2x + y^2 - \cos(x+y) &= \frac{\partial F}{\partial x} \\ 2xy - \cos(x+y) - e^{y} &= \frac{\partial F}{\partial y} \end{aligned}\right\} \Rightarrow F = -\sin(x+y) - e^{y} + x^2 + xy^2 + C$$

Thus the sol^{n} is given implicitly by $\boxed{-\sin(x+y) - e^{y} + x^2 + xy^2 + C = 0}$

Remark: We now have 3-types of 1st order ODEs we can solve. ① seperable ② Standard Form ③ Exact. Are standard form DEq^{n} exact?

$$\frac{dy}{dx} + P(x)y = Q(x) \longrightarrow dy + (P(x)y - Q(x))\,dx = 0$$

Identify then that $M = P(x)y - Q(x)$ & $N = 1$, $\frac{\partial M}{\partial y} = P(x)$ yet $\frac{\partial N}{\partial y} = 0$ $\Rightarrow$ standard form DEq^{n} is inexact in general. We can solve ② using the integrating factor, we solve ③ by finding F whose level surfaces implicit sol^{n}'s.

§2.4 #22 $\left[ye^{xy} - \frac{1}{y}\right]dx + \left[xe^{xy} + \frac{x}{y^2}\right]dy$

From #16 we found $e^{xy} - x/y + C = 0$ now use $y(1) = 1$ to determine C. $e^1 - 1/1 + C = 0 \Rightarrow C = 1 - e$ $\therefore$ $\boxed{e^{xy} - \frac{x}{y} + 1 - e = 0}$

§2.4 #23 $\underbrace{[e^t y + te^t y]}_{M = \frac{\partial F}{\partial t}}dt + \underbrace{[te^t + 2]}_{N = \frac{\partial F}{\partial y}}dy = 0$ $\left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \text{ it's exact.}\right)$

Find then $F(t, y)$ such that $M = F_t$ and $N = F_y$.

$$F = \int \frac{\partial F}{\partial y} dy = \int (te^t + 2)dy = y(te^t + 2) + C_1(t) \qquad (*)$$

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial t}\left(y(te^t + 2) + C_1(t)\right) \quad \text{using } (*)$$

$$= y(\cancel{e^t + te^t}) + \frac{\partial C_1}{\partial t} = M = (\cancel{e^t + te^t})y$$

Hence $\frac{\partial C_1}{\partial t} = 0$ and $C_1 = C_1(t) \Rightarrow C_1 = C \leftarrow$ just a plain-old constant.

Thus $F = y(te^t + 2) + C = 0$ gives general implicit sol$^=$
however here we also know $y(0) = -1$ (from problem statement)

$$-1(0e^0 + 2) + C = 0 \Rightarrow C = 2 \Rightarrow \boxed{y(te^t + 2) + 2 = 0}$$

Or explicitly; $\boxed{y = \frac{-2}{te^t + 2}}$

§2.4 #32 Consider family of curves given implicitly by $F(x, y) = k$

(a.) Curves in this family (call it C_k) have slope given by

$$\frac{dy}{dx} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial y}$$

orthogonal trajectories will satisfy $\left.\frac{dy}{dx}\right|_{OT} = \frac{-1}{\left(\left.\frac{dy}{dx}\right|_{C_k}\right)}$
that means that

$$\left.\frac{dy}{dx}\right|_{OT} = \frac{\partial F}{\partial y} \Big/ \frac{\partial F}{\partial x},$$

For the O.T then $\frac{\partial F}{\partial x}dy = \frac{\partial F}{\partial y}dx \rightarrow \boxed{\frac{\partial F}{\partial y}dx - \frac{\partial F}{\partial x}dy = 0}$

Notice O.T. will not be given by $F(x,y) = k$, but rather $G(x, y) = C$ where $dG = \frac{\partial F}{\partial y}dx - \frac{\partial F}{\partial x}dy$ but we know $dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy$ thus $\boxed{\frac{\partial G}{\partial x} = \frac{\partial F}{\partial y} \;\&\; \frac{\partial G}{\partial y} = -\frac{\partial F}{\partial x}}$ *

§2.4 #32

(b.) Consider $x^2+y^2=k$ (circles) here $F(x,y)=x^2+y^2$
then construct G from (*) (cause we can.)

$$\frac{\partial G}{\partial x} = \frac{\partial F}{\partial y} = 2y \longrightarrow G = \int \frac{\partial G}{\partial x}\,dx = \int 2y\,dx = 2xy + C_1(y)$$

$$\frac{\partial G}{\partial y} = -\frac{\partial F}{\partial x} = -2x \longrightarrow \frac{\partial G}{\partial y} = 2x + \frac{\partial C_1}{\partial y} = -2x$$

Hence $\frac{\partial C_1}{\partial y} = -4x$ but $C_1 = C_1(y)$, it's impossible to find such a G. (Because for O.T. $\frac{\partial M}{\partial y} = 2$ & $\frac{\partial N}{\partial x} = -2$ it's not exact!)

ANYWAY: we can solve it w/o exact eq^{n} tricks.

For C_k $\quad x^2+y^2=k \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \therefore \frac{dy}{dx} = \frac{-x}{y}$

For O.T. $\quad \frac{dy}{dx} = \frac{y}{x} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$

$$\Rightarrow \ln|y| = \ln|x| + \tilde{c}$$
$$\Rightarrow |y| = e^{\ln|x|+\tilde{c}} = e^{\tilde{c}}|x|$$
$$\Rightarrow y = \pm e^{\tilde{c}} x$$
$$\Rightarrow \boxed{y = mx \qquad m \in \mathbb{R}}$$

orthogonal Traject.

(c.) $xy = k \Rightarrow y + x\frac{dy}{dx} = 0$ (differentiating implicitly.)

$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$

$\Rightarrow$ O.T. will satisfy $\frac{dy}{dx} = \frac{x}{y}$

$\Rightarrow$ O.T have $y\,dy = x\,dx$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + \tilde{C}$$

$$\boxed{y^2 - x^2 = C \text{ ; O.T.s.}}$$

§2.4 # 33

(a.) $d(2x^2+y^2) = 4x\,dx + 2y\,dy = dF = 0$

$\Rightarrow$ O.T. given by $2y\,dx - 4x\,dy = 0$

meaning for O.T. $\frac{dx}{x} = \frac{2dy}{y} \Rightarrow \ln|x| = 2\ln|y| + \tilde{c}$

$\Rightarrow |x| = e^{\ln|y|^2} c$

$\Rightarrow |x| = c\,|y|^2$

$\Rightarrow x = \pm c\,y^2$

$\Rightarrow \boxed{x = m y^2 \quad m \in \mathbb{R}}$

(sideways parabolas)

(b.) $y/x^4 = k \rightarrow (-4y/x^5)dx + (1/x^4)dy = 0$

$\Rightarrow$ O.T. given by $\frac{1}{x^4}dx + \frac{4y}{x^5}dy = 0$

that is $x\,dx = -4y\,dy \rightarrow \frac{1}{2}x^2 = -2y^2 + c$

$\rightarrow \boxed{\frac{x^2}{4} + \frac{y^2}{2} = \tilde{c}}$ (ellipse.)

(c.) $y = e^{kx} \Rightarrow \ln(y) = kx \Rightarrow \frac{\ln(y)}{x} = k = F$

thus $dF = \frac{-\ln(y)}{x^2}dx + \frac{1}{xy}dy = 0$ then O.T. satisfy

$$\frac{1}{xy}dx + \frac{\ln(y)}{x^2}dy = 0$$

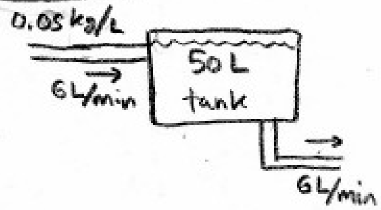
$$-x\,dx = y\ln(y)\,dy$$

$$-\frac{1}{2}x^2 + c = \frac{1}{2}y^2\ln(y) - \int \frac{y^2}{2}\frac{1}{y}dy = \frac{1}{2}y^2\ln(y) - \frac{1}{4}y^2$$

Thus O.T. given implicitly by $\boxed{y^2\left(\ln(y) - \frac{1}{2}\right) + x^2 = \tilde{c}}$

Remark: I hope you can see that my method in #32 is really the same as what I've done here. I tend to use the first method, but the one used here in #33 saves some writing.

§ 3.2 #2



Let y = mass of salt in tank at time t.

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$
$$= \left(6\,\frac{L}{min}\right)\left(0.05\,\frac{kg}{L}\right) - \frac{y}{50\,L}\cdot\frac{6\,L}{min}$$
$$= 0.3\,\frac{kg}{min} - \left(0.12\,\frac{1}{min}\right)y$$

Dropping units, solve $\frac{dy}{dt} = 0.3 - 0.12y$ given $y(0) = 0.5$

Use integrating factor,

$$\frac{dy}{dt} + 0.12y = 0.3 \Rightarrow \mu(t) = \exp\left(\int 0.12\,dt\right) = e^{0.12t}$$

$$\underbrace{e^{0.12t}\frac{dy}{dt} + 0.12e^{0.12t}y}_{} = 0.3e^{0.12t}$$

$$\underbrace{\int \frac{d}{dt}\left(e^{0.12t}y\right)dt}_{\parallel\ (FTC)} = \int (0.3)e^{0.12t}\,dt = \frac{0.3}{0.12}e^{0.12t} + C$$

$$e^{0.12t}y = \frac{4}{100}e^{0.12t} + C$$

$$y(0) = 0.5 \Rightarrow 0.5 = \frac{4}{100} + C$$
$$\Rightarrow 0.5 - 0.025 = C \Rightarrow \underline{C = 0.475}$$

Thus $\boxed{y(t) = \frac{4}{100} + 0.475e^{-0.12t}}$

§ 3.2 #3 Let y = volume of nitric acid in tank.

$$\frac{dy}{dt} = \begin{pmatrix}\text{rate of volume}\\ \text{of nitric acid in}\end{pmatrix} - \begin{pmatrix}\text{rate of volume}\\ \text{of nitric acid out}\end{pmatrix}$$

$$= \left(\frac{6L}{min}\cdot 0.2\,\frac{\text{acid }L}{L}\right) - \left(\frac{8L}{min}\right)\frac{y}{\text{Volume}}$$

$$= \frac{12}{10} - \frac{8y}{200-2t}$$

notice volume changes over time!

Vol. = $200 - 2t$

think about it, it's losing 2L/min so that means at 100 min. it's empty.

$$\frac{dy}{dt} + \underbrace{\left(\frac{8}{200-2t}\right)}_{P(t)}y = \underbrace{\frac{12}{10}}_{Q(t)}$$

Complete this by using $\mu(t) = \exp\left(\int \frac{8}{200-2t}\,dt\right)$.

§3.3 #1 Hot coffee at 95°C when $t=0$ cools to 80°C in 5 minutes in an ambient temperature of 21°C. Assume Newton's Law of Cooling to find time when coffee was 50°C. Newton said change in temp T ∝ to difference from ambient temp.

$$\frac{dT}{dt} = k(T - T_{room}) = k(T-21)$$

$$\frac{dT}{dt} - kT = -21k$$

could use integrating factor however undetermined coeff. is easier here

$T_{homo.} = c_1 e^{kt}$

$T_{particular} = A$

$\Rightarrow \frac{dT}{dt} - kT = -kA = -21k$

$\Rightarrow \underline{A = +21}$

$$T(t) = c_1 e^{kt} + 21$$

use givens to find c_1 & k.

$$\begin{cases} T(0) = c_1 + 21 = 95 \quad \therefore c_1 = 74 \\ T(5) = 74e^{5k} + 21 = 80 \end{cases}$$

$$74e^{5k} = 59 \Rightarrow e^{5k} = \frac{59}{74} \quad \therefore k = \frac{1}{5}\ln\left(\frac{59}{74}\right)$$

$$\boxed{T(t) = 74e^{\frac{1}{5}\ln\left(\frac{59}{74}\right)t} + 21}$$ (arbitrary time t.)

$$T(t) = 50 = 74e^{\frac{1}{5}\ln\left(\frac{59}{74}\right)t} + 21$$

$$\frac{29}{74} = e^{\frac{1}{5}\ln\left(\frac{59}{74}\right)t} \Rightarrow \ln\left(\frac{29}{74}\right) = \frac{1}{5}\ln\left(\frac{59}{74}\right)t$$

$$\Rightarrow t = \frac{5\ln(29/74)}{\ln(59/74)} = \boxed{20.68 \text{ min}}$$

§3.4#1 $m = 5\,kg$, $y_0 = 1000\,m$, $F_f = bV$ where $V = \frac{dy}{dt}$ $b=50$

Newton's 2^{nd} Law $\Rightarrow$ $5\frac{d^2y}{dt^2} = -5g - 50\frac{dy}{dt}$ & $g = 9.8$.

$$5y'' + 50y' = -49$$

- We can use chapter 4 methods which we know from ma241
- Solving this problem with material < §3.4 is inane.

$$5r^2 + 50r = 0$$
$$r^2 + 10r = 0$$
$$r(r+10) = 0 \Rightarrow r = 0,\ r = -10 \ \therefore\ Y_{homo.} = C_1 + C_2e^{-10t}$$

$Y_{part} = A$ then there is overlap with C_1 thus $Y_p = tA$ is appropriate choice, for the particular sol^n (undetermined coefficients)

$$\left.\begin{aligned} Y_p &= tA \\ Y_p' &= A \\ Y_p'' &= 0 \end{aligned}\right\} \longrightarrow \begin{aligned} 5Y_p'' + 50Y_p' &= -49 \\ 50A &= -49 \ \therefore\ A = \frac{-49}{50} \end{aligned}$$

Thus $Y(t) = C_1 + C_2e^{-10t} - \left(\frac{49}{50}\right)t$ (the general sol^n)
but our sol^n has intial conditions, $Y(0) = 1000$, $Y'(0) = 0$.

$$Y(0) = C_1 + C_2 = 1000$$
$$Y'(0) = -10C_2 - \frac{49}{50} = 0 \Rightarrow C_2 = \frac{-49}{500} = -0.098$$
$$\Rightarrow C_1 = 1000 - C_2 = 1000 + \frac{49}{500}$$
$$C_1 = 1000.098$$

Then $\boxed{y(t) = 1000.098 - 0.098e^{-10t} - 0.98t}$

Graph $y(t)$ and you'll find $t = 1020.5$ is a zero. As I've defined y when $y = 0$ that is the ground (the text puts $y = 0$ elsewhere, that's why it's eq^n of motion looks different)

$$0 = 1000.098 - 0.098\left(1 - \cancel{10t} + \frac{100t^2}{2} + \cdots\right) - \cancel{0.98t}$$
$$= 1000 - (0.098)\,50t^2 \Rightarrow t^2 = \frac{1000}{50(0.098)} = 204.1$$

(incorrect, hidden wrong assumption) $\Rightarrow \underline{t = \pm 14.28}$

- what's wrong with this attempt to find sol^n?

§ 3.4 # 5

$$ma = -mg - 10v \rightarrow y'' + \frac{10}{m}y' = -g$$
$$\rightarrow y'' + 2y' = -9.8$$
$$\rightarrow r^2 + 2r = r(r+2) = 0$$
$$\therefore\ r = 0 \ \& \ r = -2$$
$$\therefore\ Y_h = c_1 + c_2 e^{-2t}$$

Guess $Y_p = tA$ then $Y_p' = A$ & $Y_p'' = 0$ (using undet. coeff.)

$$2A = -g \Rightarrow A = -g/2 \Rightarrow Y_p = -\frac{gt}{2}$$

Thus $Y(t) = c_1 + c_2 e^{-2t} - \frac{1}{2}gt$. Use data $Y'(0) = -50\frac{m}{s}$ and $Y(0) = 500m$ (I take ground to be $Y = 0$).

$$Y(0) = c_1 + c_2 = 500$$
$$Y'(0) = -2c_2 - g/2 = -50 \Rightarrow c_2 = \frac{1}{2}(50 - g/2) = 22.6$$
$$c_1 = 500 - 22.6 = 477.4$$

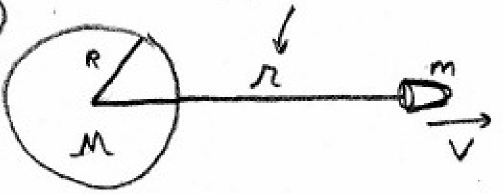
Hence $\boxed{Y(t) = 477.4 + 22.6e^{-2t} - 4.9t}$

If you graph $f(t) = 477.4 + 22.6e^{-2t} - 4.9t$ you'll find it has a zero at $t = 97.4$, that makes $Y(97.4) = 0$ and as I define Y that's the ground. (the text seems to be taking mid-air as $Y = 0$ ...)

§3.4 #25

$r = r$

r = distance from center, $r > 0$ by def^n.



$F_g = \frac{GMm}{r^2}$ in direction to attract m & M

a.) $ma = F_{net} = -\frac{GMm}{r^2} = m\frac{d^2r}{dt^2}$

divide by m $\Rightarrow -\frac{GM}{r^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{dt}(v) = \frac{dv}{dt}$

Now at the surface of the earth $r = R$ and by def^n of g we have $F_{gravity} = \cancel{m}g = \frac{G\cancel{m}M}{R^2}$ thus we find $G = gR^2/M$ then

$$\frac{dv}{dt} = -\frac{GM}{r^2} = -\frac{gR^2}{M}\frac{M}{r^2} = \boxed{-\frac{gR^2}{r^2} = \frac{dv}{dt}}$$

b.) $\frac{dv}{dt} = \frac{dr}{dt}\frac{dv}{dr}$ (assuming radial motion!)

$$= \boxed{v\frac{dv}{dr} = -\frac{gR^2}{r^2}} \quad \text{using (a.)}$$

c.) $\int_{v_0}^{v_f} v\,dv = \int_{r_0}^{r_f} -gR^2\frac{dr}{r^2} \Rightarrow \frac{1}{2}(v_f^2 - v_0^2) = gR^2\left(\frac{1}{r_f} - \frac{1}{r_0}\right)$

seperated variables and integrated from intial to final velocities & positions; note $r_0 = R$, $v_0 = v_0$, $v_f = v$ and $r_f = r$ in the langauge of the problem set up.

$$\frac{1}{2}(v^2 - v_0^2) = gR^2\left(\frac{1}{r} - \frac{1}{R}\right)$$

$$v^2 = v_0^2 + 2gR^2\left(\frac{1}{r} - \frac{1}{R}\right)$$

$$= \boxed{v_0^2 - 2gR + 2gR^2/r = v^2}$$

d.) For velocity to become negative (although strictly speaking since r = distance $\Rightarrow \frac{dr}{dt}$ = speed, but we could have taken r to be position along the direction of motion) we need $v > 0 \longrightarrow v < 0$ hence $v = 0$ for some r_e, but $KE = \frac{1}{2}mv_e^2 = 0$ for r_e notice that $\frac{1}{2}mv^2 \overset{!}{=} \frac{1}{2}m(v_0^2 - 2gR + 2gR^2/r)$

§ 3.4 # 25 Continued $\quad K.E. = \frac{1}{2}m\left(V_0^2 - 2gR + 2gR^2/r\right)$

And if V is to become negative it must be zero for some r as we assume $V > 0$ to begin. Notice that KE then will also be zero if that could happen. So how can KE be zero,

$$KE = \frac{1}{2}m\left(V_0^2 - 2gR + \frac{2gR^2}{r}\right)$$

As $r \to \infty$ notice $KE \to \frac{1}{2}m\left(V_0^2 - 2gR\right)$ if this is positive then were free from concern that $KE \to 0$ (when r is smaller that just adds to KE.)

Thus need $V_0^2 - 2gR > 0$ to insure $KE > 0$ for $r \to \infty$.

§4.2 #2

$$y''-y'-2y=0 \Rightarrow r^2-r-2=0 \Rightarrow (r-2)(r+1)=0$$
$$\Rightarrow r=-1 \text{ or } r=2 \Rightarrow \boxed{y=c_1e^{-x}+c_2e^{2x}}$$

§4.2 #4

$$y''+6y'+9y=0 \Rightarrow \lambda^2+6\lambda+9=0$$
$$\Rightarrow (\lambda+3)^2=0 \quad \therefore \lambda=-3,-3 \Rightarrow \boxed{y=c_1e^{-3x}+c_2xe^{-3x}}$$

<u>Remark:</u> I like to use λ instead of r for the variable of the characteristic eq$^{\underline{n}}$. So be warned.

§4.2 #6

$$y''-5y'+6y=0 \Rightarrow \lambda^2-5\lambda+6=(\lambda-3)(\lambda-2)=0$$
$$\therefore \lambda=3 \text{ or } 2 \Rightarrow \boxed{y=c_1e^{3x}+c_2e^{2x}}$$

§4.2 #8

$$6y''+y'-2y=0 \Rightarrow 6\lambda^2+\lambda-2=0$$
$$\Rightarrow \lambda=\frac{-1\pm\sqrt{1-4(6)(-2)}}{12}=\frac{-1\pm\sqrt{49}}{12}$$
$$\Rightarrow \lambda_1=\frac{-1+7}{12} \;\&\; \lambda_2=\frac{-1-7}{12}$$
$$\Rightarrow \lambda_1=1/2 \text{ and } \lambda_2=-2/3$$
$$\Rightarrow \boxed{y=c_1e^{\frac{1}{2}x}+c_2e^{-\frac{2}{3}x}}$$

§4.2 #10

$$4y''-4y'+y=0 \Rightarrow 4\lambda^2-4\lambda+1=0$$
$$\Rightarrow \lambda=\frac{4\pm\sqrt{16-16}}{8}=\frac{1}{2}=\lambda_1=\lambda_2$$
$$\Rightarrow \boxed{y=c_1e^{\frac{1}{2}x}+c_2xe^{\frac{1}{2}x}}$$

§4.2 #12

$$3y''+11y'-7y=0$$
$$3\lambda^2+11\lambda-7=0$$
$$\lambda=\frac{-11\pm\sqrt{121-4(3)(-7)}}{6}=\frac{-11\pm\sqrt{121+84}}{6}=\frac{-11\pm\sqrt{205}}{6}$$
$$\boxed{y=c_1\exp\left(\left(\frac{-11-\sqrt{205}}{6}\right)x\right)+c_2\exp\left(\left(\frac{-11+\sqrt{205}}{6}\right)x\right)}$$

§4.2 #14 $\quad y'' + y' = 0$ with $y(0) = 2$ and $y'(0) = 1$.

$$\lambda^2 + \lambda = \lambda(\lambda+1) = 0 \quad \therefore \lambda_1 = 0,\ \lambda_2 = -1 \quad \therefore y = c_1 + c_2 e^{-x}$$

using data given
$$\begin{cases} y(0) = c_1 + c_2 e^0 = 2 \\ y'(0) = -c_2 e^0 = 1 \end{cases} \Rightarrow c_2 = -1 \quad \& \quad c_1 = 2 - c_2 = 3.$$

Thus $\boxed{y = 3 - e^{-x}}$

§4.2 #18 $\quad y'' - 6y' + 9y = 0$

$$\lambda^2 - 6\lambda + 9 = (\lambda-3)^2 = 0 \quad \therefore \lambda_1 = \lambda_2 = 3.$$

$$y = c_1 e^{3x} + c_2 x e^{3x}$$

$$y' = 3c_1 e^{3x} + c_2(e^{3x} + 3x e^{3x}) \qquad \text{(product rule, remember.)}$$

using givens.
$$\begin{cases} y(0) = c_1 = 2 \\ y'(0) = 3c_1 + c_2 = 6 + c_2 = 25/3 \end{cases} \Rightarrow 3c_2 = 25 - 18 = 7 \Rightarrow c_2 = 7/3$$

$$\boxed{y = 2e^{3x} + \frac{7}{3} x e^{3x}}$$

§4.2 #28 $\quad y_1(t) = e^{3t}$ and $y_2(t) = e^{-4t}$ can we find some constant c such that $y_1 = c\,y_2$, that is $e^{3t} = ce^{-4t}$?
Suppose we could then this constant c would have the following rather curious property

$$e^{3t} = ce^{-4t} \Rightarrow e^{7t} = c \Rightarrow$$ the "constant" depends on. t, thus $\nexists$ any such constant hence y_1 & y_2 are L.I. (Linearly Independent)

§4.2 #31 $\quad y_1 = \tan^2 t - \sec^2 t$
$y_2 = 3$

At first glance looks L.I, but $\tan^2 t - \sec^2 t = 1 \ \therefore y_1 = 1$ and
Thus $y_2 = 3y_1$ therefore y_1 & y_2 are linearly dependent.

§4.2 #29 $\quad y_1 = te^{2t}$
$y_2 = e^{2t}$

Assume $y_1 = c\,y_2$ and seek a $\rightarrow\!\leftarrow$ (contradiction)

$$te^{2t} = ce^{2t}$$

$\Rightarrow t = c \Rightarrow$ c not a constant! but c is a constant $\rightarrow\!\leftarrow$.
Therefore $\nexists$ any such c establishing that y_1 & y_2 are LI.

§4.2 # 34 WRONSKIAN (BABY-VERSION, see chapter 6 for more)

a.) $\begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix} \overset{\text{def}^n}{\equiv} Y_1 Y_2' - Y_1' Y_2 \overset{\text{def}^n}{\equiv} W[Y_1, Y_2]$ (suppressing t-dependence)

def^n of determinant of 2×2 matrix

b.) Suppose $aY'' + bY' + cY = 0$ has sol^n's $Y_1(t)$ and $Y_2(t)$

- Prove that Y_1 & Y_2 are L.I. on an interval I if and only if $W[Y_1, Y_2](t) \neq 0 \ \forall t \in I$. Consider,

Suppose, $Y_1(t)Y_2'(t) - Y_1'(t)Y_2(t) = W[Y_1, Y_2](t) = f(t) \neq 0 \quad \forall t \in I$.

Now assume $Y_2(t) = cY_1(t)$ towards a contradiction. Notice

$$Y_2'(t) = c\,Y_1'(t)$$

Thus $W[Y_1, Y_2](t) = \cancel{Y_1(t)\,c\,Y_1'(t)} - \cancel{Y_1'(t)\,c\,Y_1(t)} = 0$ but this contradicts that $W[Y_1, Y_2](t) \neq 0$.

Next, suppose that Y_1 and Y_2 are L.I. Then suppose towards a contradiction that $W[Y_1, Y_2](t) = 0$ then we have,

$$Y_1(t)Y_2'(t) - Y_1'(t)Y_2(t) = 0$$

Now since Y_1 & Y_2 are LI it follows that they cannot be the zero functions, that means we'll can find some small nbhd where $Y_1(t) \neq 0 \quad \forall t \in J$. (so we can ÷ by Y_1 on J)

Consider then that

$$\frac{d}{dt}\left[\frac{Y_2}{Y_1}\right] = \frac{Y_2'Y_1 - Y_2Y_1'}{Y_1^2} = \frac{0}{Y_1^2} = 0$$

(holds for all t on J where ÷ by $Y_1(t)$ makes sense)

Thus on J we find $\frac{Y_2}{Y_1} = c \Rightarrow Y_2 = cY_1$.

But we need $Y_2(t) = cY_1(t) \ \forall t \in I$ (J is contained in I).

We can extend to I because if $Y_2 = cY_1 \ \forall t \in J \Rightarrow$ they share same intial conditions, then we know sol^n's to $aY'' + bY' + cY = 0$ are uniquely determined by particular intial conditions. Hence $Y_2 = cY_1$ on I therefore Y_2 is not Linearly independent of Y_1 a contradiction //

(assuming Y_1 & Y_2 are sol^n's to some 2nd order ODE)

$\boxed{\text{Thus } W[Y_1, Y_2](t) \neq 0 \ \forall t \in I \iff Y_1 \text{ \& } Y_2 \text{ L.I. on } I}$

§4.3 #2 $\quad y'' + y = 0$

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow \boxed{y = c_1 \cos(x) + c_2 \sin(x)}$

§4.3 #4 $\quad y'' - 10y' + 26y = 0$

$\lambda^2 - 10\lambda + 26 = 0 \Rightarrow \lambda = \dfrac{10 \pm \sqrt{100 - 4(26)}}{2} = 5 \pm \dfrac{\sqrt{-16}}{2} = 5 \pm 2i$

Identify $\alpha = 5$ and $\beta = 2 \Rightarrow \boxed{y = e^{5x}(c_1 \cos 2x + c_2 \sin 2x)}$

§4.3 #8 $\quad 4y'' + 4y' + 6y = 0$

$4\lambda^2 + 4\lambda + 6 = 0 \Rightarrow \lambda = \dfrac{-4 \pm \sqrt{16 - 4(4)6}}{8} = \dfrac{-4 \pm \sqrt{-80}}{8} = \underbrace{\dfrac{-1}{2}}_{\alpha} \pm i \underbrace{\sqrt{\dfrac{80}{64}}}_{\beta}$

$\sqrt{\dfrac{80}{64}} = \sqrt{\dfrac{10}{8}} = \sqrt{\dfrac{5}{4}} = \dfrac{\sqrt{5}}{2} = \beta \Rightarrow \boxed{y = e^{-x/2}\left(c_1 \cos\left(\tfrac{\sqrt{5}}{2}x\right) + c_2 \sin\left(\tfrac{\sqrt{5}}{2}x\right)\right)}$

Remark: could also write:

$$y = Ae^{-x/2} \sin\left(\tfrac{\sqrt{5}}{2}x + \varphi\right)$$

exercise, relate c_1 & c_2 to A & φ.

§4.3 #10 $\quad y'' + 4y' + 8y = 0$

$\lambda^2 + 4\lambda + 8 = 0 \Rightarrow \lambda = \dfrac{-4 \pm \sqrt{16 - 32}}{2} = -2 \pm 2i = \alpha \pm i\beta.$

$\boxed{y = e^{-2x}(c_1 \cos(2x) + c_2 \sin(2x))}$

§4.3 #12 $\quad u'' + 7u = 0$

$\lambda^2 + 7 = 0 \Rightarrow \lambda^2 = -7 \Rightarrow \lambda = \pm\sqrt{-7} = \pm 7i \qquad (\alpha = 0,\ \beta = 7)$

$\boxed{u = c_1 \cos 7x + c_2 \sin 7x}$

§4.3 #22 $\quad y'' + 2y' + 17y = 0$

$\lambda^2 + 2\lambda + 17 = 0 \rightarrow \lambda = \dfrac{-2 \pm \sqrt{4 - 4(17)}}{2} = -1 \pm 4i = \alpha \pm i\beta$ *

$* \Rightarrow y = e^{-x}(c_1 \cos 4x + c_2 \sin 4x)$

$y' = -e^{-x}(c_1 \cos 4x + c_2 \sin 4x) + e^{-x}(-4c_1 \sin 4x + 4c_2 \cos 4x)$ (product rule)

using the given conditions
$\begin{cases} y(0) = c_1 = 1 \\ y'(0) = -c_1 + 4c_2 = -1 \Rightarrow 4c_2 = 0 \Rightarrow c_2 = 0 \end{cases}$ $\therefore \boxed{y = e^{-x} \cos 4x}$

§4.4 #1 no t^{-1} is not allowed, derivatives of t^{-1} are unending!

§4.4 #2 sure.

§4.4 #3 yep, just remember $3^t = e^{\ln(3^t)} = e^{\ln(3)t}$.

§4.4 #4 yep $\sin(x)/e^{4x} = e^{-4x}\sin(x)$, derivatives close back on themselves,

Remark: When solving $ay'' + by' + cy = g(x)$ the method of undetermined coefficients will work so long as the function $g(x)$ and it's derivatives $g'(x), g''(x), \dots$ form a finite set of functions upto linear independence. The text explains the possibilities more explicitly as you should discover.

§4.4 #8 sure, but it would be horrible.

§4.4 #9 $y'' + 3y = -9$

$\lambda^2 + 3 = 0 \Rightarrow \lambda = \pm\sqrt{3}\,i \Rightarrow y_h = c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)$

Clearly there is no overlap of y_1 and y_2 with $g(x)$ thus guess $y_p = A$

$y_p'' + 3y_p = -9 \Rightarrow 3A = -9 \Rightarrow \underline{A = -3} \quad \therefore \boxed{y_p = -3}$

The general sol$^{\circ}$ would then be $y = y_h + y_p$.

§4.4 #12 $2x' + x = 3t^2 \Rightarrow y_p = At^2 + Bt + C$

$2\lambda + 1 = 0 \Rightarrow \lambda = -1/2 \Rightarrow y_h = c_1 e^{-\frac{1}{2}t}$ (no overlap with y_p.)

Substituting x_p into DEq^{n}: $2x_p' + x_p = 2(2At + B) + At^2 + Bt + C = 3t^2$

$$\underline{At^2 + (4A + B)t + (2B + C) = 3t^2} \quad (*)$$

Equate Coefficients of like powers of t in $(*)$ to get 3 eq^{n}'s below,

$t^2 : A = 3$

$t^1 : 4A + B = 0 \Rightarrow B = -12$

$t^0 : 2B + C = 0 \Rightarrow C = 24 \quad \therefore \boxed{x_p = 3t^2 - 12t + 24}$

§4.4 #14 $y'' + y = 2^x = e^{\ln(2)\cdot x}$

Note $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow y_h = c_1\cos(x) + c_2\sin(x)$.

Thus no overlap with y_p (naive), simply use $y_p = Ae^{\ln(2)x} = A2^x$

$$\left.\begin{aligned} y_p' &= \ln(2)\,A2^x \\ y_p'' &= (\ln(2))^2 A2^x \end{aligned}\right\} \Rightarrow (\ln(2))^2 A2^x + A2^x = 2^x$$

$$A([\ln(2)]^2 + 1)\,2^x = 2^x$$

$$\Rightarrow A = \frac{1}{[\ln(2)]^2 + 1}$$

$$\therefore \boxed{y_p = \frac{1}{(\ln(2))^2 + 1}\,2^x}$$

§4.4 #16 $\Theta'' - \Theta = t\sin t$

$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \Rightarrow \Theta_h = c_1 e^t + c_2 e^{-t} \quad \text{(no overlap here)}$$

$$\Theta_p = t(A\sin t + B\cos t) + C\sin t + D\cos t \quad \text{(follows from } t\sin t \text{ term)}$$

$$\Theta_p' = A\sin t + B\cos t + t(A\cos t - B\sin t) + C\cos t - D\sin t \quad \text{(used prod. rule)}$$
$$= \sin t[A-D] + \cos t[B+C] + t[A\cos t - B\sin t]$$

$$\Theta_p'' = \cos t[A-D] - \sin t[B+C] + [A\cos t - B\sin t] + t[-A\sin t - B\cos t]$$
$$= \cos t[A-D+A] + \sin t[-B-C-B] + t[-A\sin t - B\cos t]$$

Now substitute Θ_p, Θ_p' & Θ_p'' into $\Theta_p'' - \Theta_p = t\sin t$ to obtain,

$$\Theta_p'' - \Theta_p = t\sin t = \cos t[2A-2D] + \sin t[-2B-2C] + t\cos t[-B-B] + t\sin t[-A-A]$$

Note the functions $t\sin t$, $t\cos t$, $\sin t$, $\cos t$ are L.I. so we can equate coefficients to find,

$$\left.\begin{array}{llll} t\sin t: & 1 = -2A & \Rightarrow & A = -1/2 \\ t\cos t: & 0 = -2B & \Rightarrow & B = 0 \\ \sin t: & 0 = -2B - 2C & \Rightarrow & C = 0 \\ \cos t: & 0 = 2A - 2D & \Rightarrow & D = -1/2 \end{array}\right\} \quad \boxed{\Theta_p = -\tfrac{1}{2}t\sin t - \tfrac{1}{2}\cos t}$$

§4.4 # 36 Find particular sol^n to $y'''' - 3y'' - 8y = \sin(t)$. Really should find Y_h to insure no overlap. Here the characteristic eq^n will be 4th order $\Rightarrow$ 4 complex sol^n's.

$$\lambda^4 - 3\lambda^2 - 8 = 0 \quad \text{let } \lambda^2 = \delta \text{ to reduce to quadratic,}$$

$$\delta^2 - 3\delta - 8 = 0 \Rightarrow \delta = \frac{3 \pm \sqrt{9+4}}{2} = \frac{3 \pm \sqrt{13}}{2}$$

Hence $\delta = \lambda^2 = \frac{3}{2} \pm \frac{\sqrt{13}}{2} \Rightarrow \lambda = \pm\sqrt{\frac{3}{2} \pm \frac{\sqrt{13}}{2}}$ where the $\pm$ are not connected but instead give 4 outcomes.

$$\lambda_1 = \sqrt{\frac{3+\sqrt{13}}{2}}, \quad \lambda_2 = \sqrt{\frac{3-\sqrt{13}}{2}}, \quad \lambda_3 = -\lambda_1 \text{ and } \lambda_4 = -\lambda_2$$

Thus the auxillary or homogeneous sol^n would be

$$Y_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t} + c_4 e^{\lambda_4 t} \quad \text{(no overlap)}$$

Now we find Y_p as usual, guess $Y_p = A\sin t + B\cos t$

$$\left.\begin{array}{l} Y_p' = A\cos t - B\sin t \\ Y_p'' = -A\sin t - B\cos t = -Y_p \\ Y_p''' = -Y_p' = -A\cos t + B\sin t \\ Y_p'''' = -Y_p'' = -(-Y_p) = Y_p \end{array}\right\} \quad \begin{array}{l} Y_p'''' - 3Y_p'' - 8Y_p = \sin t \\ Y_p - 3(-Y_p) - 8Y_p = 12Y_p = \sin t \\ 12A\sin t + 12B\cos t = \sin t \end{array}$$

Comparing coefficients of $\cos t$ and $\sin t$ yields $12A = 1$ & $12B = 0$

Therefore, $\boxed{Y_p = \frac{1}{12}\sin(t)}$ The general sol^n would be $Y = Y_h + Y_p$.

§4.5 #1 First I'd like to say I'm a bit annoyed the book has used Y_1 & Y_2 for particular sol^{n}s instead lets say

$$Y_{P_1} = \cos t \quad \text{and} \quad Y_{P_2} = e^{2t}/3$$

These satisfy $y'' - y' + y = \sin t$ or $y'' - y' + y = e^{2t}$ respective. I reserve Y_1 & Y_2 for sol^{n}'s to the auxillary Deq^n $y'' - y' + y = 0$. that eq^n has sol^n of the form $Y_h = C_1 Y_1 + C_2 Y_2$ where we determine Y_1 & Y_2 through the characteristic eq^n and such,

$$\lambda^2 - \lambda + 1 = 0 \Rightarrow \lambda = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

Hence $Y_h = e^{t/2}\left(C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\sqrt{\frac{3}{2}}\,t\right)\right)$. Now lets go on to address the books questions, however we will find the general sol^n to the following

a.) $y'' - y' + y = 5 \sin t$ has $Y_p = 5 Y_{P_1} = 5 \cos t$ using Th^m (3)

hence $\boxed{Y = e^{t/2}\left(C_1 \cos\left(\sqrt{\tfrac{3}{2}}\,t\right) + C_2 \sin\left(\sqrt{\tfrac{3}{2}}\,t\right)\right) + 5\cos t}$

it is true that $Y = 5\cos t$ is a sol^n as well but the boxed sol^n accounts for all possible intial conditions whereas $Y = 5 \cos t$ is a specific case $(C_1 = C_2 = 0)$.

Now I'll behave and follow the books implicit thinking,

b.) $y'' - y' + y = \sin(t) - 3e^{2t}$

$$Y_{P_1} = \cos t \qquad A Y_{P_2} = -3e^{2t} \Rightarrow A\frac{e^{2t}}{3} = -3e^{2t} \Rightarrow A = -9$$

Hence $\boxed{Y_p = \cos t - 9e^{2t}/3}$ is a sol^n using Th^m(3) again.

c.) $y'' - y' + y = 4 \sin t + 18 e^{2t}$ $\quad \left(18e^{2t} = Ae^{2t}/3 \therefore A = 54.\right)$

$\Rightarrow \boxed{Y_p = 4\cos t + 54e^{2t}/3}$ using Th^m (3).

Remark: In (b.) and (c.) I have found a sol^n but not the most general sol^n. To do that we would have to add the homogeneous sol^n which accounts for the myriad of possible intial conditions.

§4.5#18 $y'' - 2y' - 3y = 3t^2 - 5$

As usual find homogeneous sol^{n} to begin $\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$

Hence $\lambda_1 = 3$ and $\lambda_2 = -1$ $\therefore$ $y_h = c_1 \exp(3t) + c_2 \exp(-t)$.

Assemble y_p from $3t^2 - 5 \Rightarrow$

$$y_p = At^2 + Bt + C$$
$$y_p' = 2At + B$$
$$y_p'' = 2A$$

Now substitute y_p into the DEq^{n} to determine A, B & C,

$$y_p'' - 2y_p' - 3y_p = 3t^2 - 5$$

$$2A - 2(2At + B) - 3(At^2 + Bt + C) = 3t^2 - 5$$

$$[2A - 2B - 3C] + t[-4A - 3B] + t^2[-3A] = [-5] + t[0] + t^2[3]$$

Hence comparing coefficients,

$$3 = -3A \longrightarrow A = -1$$
$$0 = -4A - 3B \longrightarrow 3B = 4 \longrightarrow B = 4/3$$
$$-5 = 2A - 2B - 3C \longrightarrow 3C = 2A - 2B + 5 \Rightarrow C = \tfrac{1}{3}\left(-2 - \tfrac{8}{3} + 5\right) = \tfrac{1}{9}$$

Thus $\boxed{y = y_h + y_p = c_1 \exp(3t) + c_2 \exp(-t) - t^2 + \frac{4}{3}t + \frac{1}{9}}$ (general sol^{n})

§4.5#22 $y'' + 6y' + 10y = 10x^4 + 24x^3 + 2x^2 - 12x + 18 \equiv g(x)$

Auxillary sol^{n} $\lambda^2 + 6\lambda + 10 = 0 \Rightarrow \lambda = \frac{-6 \pm \sqrt{36 - 40}}{2} = -3 \pm i \Rightarrow y_h = e^{-3x}(c_1 \cos(x) + c_2 \sin(x))$.

Guess:

$$y_p = Ax^4 + Bx^3 + Cx^2 + Dx + E$$
$$y_p' = 4Ax^3 + 3Bx^2 + 2Cx + D$$
$$y_p'' = 12Ax^2 + 6Bx + 2C$$

Hence substituting into $y_p'' + 6y_p' + 10y_p = 10x^4 + 24x^3 + 2x^2 - 12x + 18$

$$g(x) = (12Ax^2 + 6Bx + 2C) + $$
$$+ 6(4Ax^3 + 3Bx^2 + 2Cx + D) + $$
$$+ 10(Ax^4 + Bx^3 + Cx^2 + Dx + E)$$

(*)
$$= x^4[10A] + x^3[10B + 24A] + x^2[10C + 18B + 12A] + x[10D + 12C + 6B] + [10E + 6D + 2C]$$
$$= g(x) = x^4[10] + x^3[24] + x^2[2] + x[-12] + [18]$$

§4.5 #18 $y'' - 2y' - 3y = 3t^2 - 5$

As usual find homogeneous sol$^{\underline{n}}$ to begin $\lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1) = 0$

Hence $\lambda_1 = 3$ and $\lambda_2 = -1$ $\therefore$ $y_h = c_1 \exp(3t) + c_2 \exp(-t)$.

Assemble y_p from $3t^2 - 5 \Rightarrow$

$$y_p = At^2 + Bt + C$$
$$y_p' = 2At + B$$
$$y_p'' = 2A$$

Now substitute y_p into the DEq$^{\underline{n}}$ to determine A, B & C,

$$y_p'' - 2y_p' - 3y_p = 3t^2 - 5$$

$$2A - 2(2At + B) - 3(At^2 + Bt + C) = 3t^2 - 5$$

$$[2A - 2B - 3C] + t[-4A - 3B] + t^2[-3A] = [-5] + t[0] + t^2[3]$$

Hence comparing coefficients,

$$3 = -3A \longrightarrow A = -1$$
$$0 = -4A - 3B \longrightarrow 3B = 4 \longrightarrow B = 4/3$$
$$-5 = 2A - 2B - 3C \longrightarrow 3C = 2A - 2B + 5 \Rightarrow C = \tfrac{1}{3}\left(-2 - \tfrac{8}{3} + 5\right) = \tfrac{1}{9}$$

Thus $\boxed{y = y_h + y_p = c_1 \exp(3t) + c_2 \exp(-t) - t^2 + \frac{4}{3}t + \frac{1}{9}}$ (general sol$^{\underline{n}}$)

§4.5 #22 $y'' + 6y' + 10y = 10x^4 + 24x^3 + 2x^2 - 12x + 18 \equiv g(x)$

Auxillary sol$^{\underline{n}}$ $\lambda^2 + 6\lambda + 10 = 0 \Rightarrow \lambda = \frac{-6 \pm \sqrt{36-40}}{2} = -3 \pm i \Rightarrow y_h = e^{-3x}(c_1\cos(x) + c_2\sin(x))$.

Guess: $y_p = Ax^4 + Bx^3 + Cx^2 + Dx + E$

$$y_p' = 4Ax^3 + 3Bx^2 + 2Cx + D$$
$$y_p'' = 12Ax^2 + 6Bx + 2C$$

Hence substituting into $y_p'' + 6y_p' + 10y_p = 10x^4 + 24x^3 + 2x^2 - 12x + 18$,

$$g(x) = (12Ax^2 + 6Bx + 2C) + 6(4Ax^3 + 3Bx^2 + 2Cx + D) + 10(Ax^4 + Bx^3 + Cx^2 + Dx + E)$$

$$(*) \begin{cases} = x^4[10A] + x^3[10B + 24A] + x^2[10C + 18B + 12A] + x[10D + 12C + 6B] + [10E + 6D + 2C] \\ = g(x) = x^4[10] + x^3[24] + x^2[2] + x[-12] + [18] \end{cases}$$

§4.5#22 Continued, compare coefficients of (*) work from x^4 down,

$$10 = 10A \longrightarrow A = 1$$
$$24 = 24A + 10B \longrightarrow B = 0$$
$$2 = 10C + 18B + 12A \rightarrow C = -1$$
$$-12 = 10D + 12C + 6B \rightarrow D = 0$$
$$18 = 10E + 6D + 2C \rightarrow E = 20$$

In each level I use what I learned in the last. Notice this would be much harder if you went from constants to x^4 instead.

Hence $\boxed{y = y_h + y_p = e^{-3x}(c_1 \cos x + c_2 \sin x) + x^4 - x^2 + 20}$

§4.5 #23 $y' - y = 1$ with $y(0) = 0$, find sol^n. We could use the integrating factor method, but the following is easier,

$$\lambda - 1 = 0 \Rightarrow \lambda = 1 \Rightarrow y_h = c_1 \exp(t)$$, no overlap with 1.

Hence, $y_p = A$ then $y_p' = 0$ so $y_p' - y_p = -A = 1 \therefore \underline{A = -1}$

Thus $y = c_1 \exp(t) - 1$. (the general sol^n, btw it is the general sol^n you must fit to intial data because fitting the homogeneous sol^n alone would be nonsense.)

$$y(0) = 0 = c_1 - 1 \therefore c_1 = 1 \therefore \boxed{y = \exp(t) - 1}$$

§4.5#26 $y'' + 9y = 27 \Rightarrow y_h = c_1 \cos 3x + c_2 \sin(3x) \Rightarrow$ no overlap.

Guess $y_p = A$ then $y_p' = y_p'' = 0$ thus $9A = 27 \Rightarrow A = 3$.

Giving general sol^n $\underline{y = c_1 \cos(3x) + c_2 \sin(3x) + 3}$

givens $\begin{cases} y(0) = 4 = c_1 + 3 \Rightarrow c_1 = 1 \\ y'(0) = 6 = 3c_2 \Rightarrow c_2 = 2 \end{cases}$

Thus, $\boxed{y = \cos(3x) + 2\sin(3x) + 3}$

Remark: adding intial conditions is straight forward, all we need to do is to use the data to fit c_1 and c_2 (or in general for a n^{th}-order problem $c_1, c_2, \dots, c_n$).

§4.6 #2 $y'' + y = \sec t$; note $y_h = c_1 \cos t + c_2 \sin t$. That is we have fundamental sol^{n}'s $y_1 = \cos t$ and $y_2 = \sin t$. Now then the method of variation of parameters begins with the ansatz,

$$y_p = V_1 y_1 + V_2 y_2$$

Here V_1 and V_2 play the role A, B, C, ... played in the method of undetermined coefficients, big difference here is that V_1 & V_2 are unknown functions of t which we are to determine. Upon finding them we'll have a particular sol^{n} that we could not have found with previous methods. As derived on p. 194–195 of text we see that the conditions that V_1 & V_2 must meet for $Y_p = V_1 y_1 + V_2 y_2$ to solve $y'' + y = \sec t = g(t)$ are,

$$\left.\begin{array}{l} y_1 V_1' + y_2 V_2' = 0 \\ y_1' V_1' + y_2' V_2' = g/a \end{array}\right\} \longrightarrow \boxed{\begin{array}{ll} \cos t\, V_1' + \sin t\, V_2' = 0 & (1) \\ -\sin t\, V_1' + \cos t\, V_2' = \sec(t) = \frac{1}{\cos t} & (2) \end{array}}$$

Solve (2) for $V_2' = \frac{\sin t}{\cos t} V_1' + \frac{1}{\cos^2 t}$ now substitute into Eq^{n} (1),

$$\cos t\, V_1' + \sin t \left(\frac{\sin t}{\cos t} V_1' + \frac{1}{\cos^2 t} \right) = 0$$

$$V_1' \left(\cos t + \frac{\sin^2 t}{\cos t} \right) = \frac{-\sin t}{\cos^2 t} \Rightarrow V_1' (\cos^2 t + \sin^2 t) = -\frac{\sin t}{\cos t}$$

$$\Rightarrow \frac{dV_1}{dt} = \frac{-\sin t}{\cos t} \to dV_1 = \frac{-\sin t\, dt}{\cos t}$$

$$dV_1 = \frac{du}{u}$$

$$\Rightarrow V_1 = \ln|\cos t|$$

Now since $V_1' = -\frac{\sin t}{\cos t}$ we find for V_2' that

$$V_2' = \frac{\sin t}{\cos t} \left(\frac{-\sin t}{\cos t} \right) + \frac{1}{\cos^2 t} = \frac{1 - \sin^2 t}{\cos^2 t} = \frac{\cos^2 t}{\cos^2 t} = 1 = \frac{dV_2}{dt}$$

$$\Rightarrow V_2 = t$$

Hence $y_p = \cos t \ln|\cos t| + t \sin t$

$$\Rightarrow y = y_h + y_p = \boxed{c_1 \cos t + c_2 \sin t + \cos t \ln|\cos t| + t \sin t}$$

general sol^{n}.

§4.6 #3 Consider $2x'' - 2x' - 4x = 2e^{3t}$

To begin ÷ by 2, $x'' - x' - 2x = e^{3t}$ now proceed,

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0 \Rightarrow x_1 = e^{2t} \text{ \& } x_2 = e^{-t}$$

Solve then, in the goal of finding $x_p = v_1 x_1 + v_2 x_2$,

$$v_1' x_1 + v_2' x_2 = 0 \qquad v_1' e^{2t} + v_2' e^{-t} = 0 \quad (1)$$
$$v_1' x_1' + v_2' x_2' = e^{3t} \qquad 2v_1' e^{2t} - v_2' e^{-t} = e^{3t} \quad (2)$$

now we must solve these for v_1 & v_2.

Notice $(1) \Rightarrow v_2' = -v_1' e^{3t}$ thus (2) becomes

$$2v_1' e^{2t} - (-v_1' e^{3t}) e^{-t} = e^{3t}$$
$$\Rightarrow 3v_1' e^{2t} = e^{3t}$$
$$\Rightarrow v_1' = \tfrac{1}{3} e^{t} \Rightarrow \boxed{v_1 = \tfrac{1}{3} e^{t}}$$

Then $v_2' = -v_1' e^{3t} = -\frac{1}{3} e^{t} e^{3t} = \frac{-1}{3} e^{4t} = v_2'$

$$\Rightarrow \boxed{v_2 = \tfrac{-1}{12} e^{4t}}$$

Thus
$$x = x_h + x_p$$
$$= c_1 e^{2t} + c_2 e^{-t} + \tfrac{1}{3} e^{t} e^{2t} - \tfrac{1}{12} e^{4t} e^{-t}$$
$$= \boxed{c_1 e^{2t} + c_2 e^{-t} + \tfrac{1}{4} e^{3t}}$$

<u>Remark</u>: this is a good example of what <u>not</u> to do. We can solve this problem more efficiently thru undet. coeff.

$$\left.\begin{aligned} x_p &= Ae^{3t} \\ x_p' &= 3Ae^{3t} \\ x_p'' &= 9Ae^{3t} \end{aligned}\right\} \qquad \begin{aligned} x_p'' - x_p' - 2x_p &= e^{3t} \\ (9A - 3A - 2A)e^{3t} &= e^{3t} \\ 4A &= 1 \quad \therefore A = \tfrac{1}{4} \end{aligned}$$

$$\therefore x_p = \tfrac{1}{4} e^{3t}$$

<u>Point</u>: variation of parameters is a weapon of last resort.

§4.6 #7 $y'' + 16y = \sec 4\theta$

$y_1 = \cos 4\theta$ & $y_2 = \sin 4\theta$ not hard to see.

We suppose that $Y_p = V_1 y_1 + V_2 y_2$ thus solve,

$$V_1' y_1 + V_2' y_2 = 0$$
$$V_1' y_1' + V_2' y_2' = \sec 4\theta$$

These become,

$$V_1' \cos 4\theta + V_2' \sin 4\theta = 0 \qquad :(1)$$
$$-4V_1' \sin 4\theta + 4V_2' \cos 4\theta = \sec 4\theta = \frac{1}{\cos 4\theta} \qquad :(2)$$

Eq^{n} (1) says $V_1' = -V_2' \tan 4\theta$ thus (2) becomes,

$$-4(-V_2' \tan 4\theta) \sin 4\theta + 4V_2' \cos 4\theta = \frac{1}{\cos 4\theta}$$

$$V_2' \left(\frac{\sin^2 4\theta}{\cos 4\theta} + \cos 4\theta \right) = \frac{1}{4 \cos 4\theta}$$ multiply by $\cos 4\theta$

$$V_2' (\sin^2 4\theta + \cos^2 4\theta) = 1/4 \Rightarrow \frac{dV_2}{d\theta} = 1/4 \therefore \boxed{V_2 = \theta/4}$$

Consider then (1) now that we know $V_2' = 1/4$ it simplifies,

$$V_1' \cos 4\theta + \tfrac{1}{4} \sin 4\theta = 0 \Rightarrow V_1' = \frac{-1}{4} \frac{\sin 4\theta}{\cos 4\theta}$$

$$V_1 = \int \frac{dV_1}{d\theta} d\theta = \int \frac{-1}{4} \frac{\sin 4\theta}{\cos 4\theta} d\theta = \frac{1}{16} \int \frac{du}{u} = \frac{1}{16} \ln|\cos 4\theta| = V_1$$

Thus $Y = Y_h + Y_p = \boxed{c_1 \cos 4\theta + c_2 \sin 4\theta + \frac{1}{16} \cos 4\theta \ln|\cos 4\theta| + \frac{\theta}{4} \sin 4\theta}$

general sol^{n}.

using (10) instead

$$V_1 = \int \frac{-g(t) Y_2 \, d\theta}{a[Y_1 Y_2' - Y_1' Y_2]} = \int \frac{-\sec 4\theta \sin 4\theta \, d\theta}{\cos 4\theta \cdot 4\cos 4\theta + 4 \sin 4\theta \sin 4\theta} = \int -\frac{\tan 4\theta}{4} d\theta = \frac{1}{16} \ln|\cos 4\theta|$$

$$V_2 = \int \frac{g(t) Y_1 \, d\theta}{Y_1 Y_2' - Y_1' Y_2} = \int \frac{\sec 4\theta \cos 4\theta}{4} d\theta = \int \frac{d\theta}{4} = \frac{\theta}{4}$$

Clearly Eq^{n} (10) tends to simplify these calculations, the sentence at the bottom of p. 195 is questionable. Here's the rub though, these are not as easy to memorize as Eq^{n} (9). And besides to be really pure about working from base principles you'd start with $Y_p = V_1 y_1 + V_2 y_2$ and derive eq^{n} (9). I'll allow you to memorize the eq^{n} (10) for more efficient calculation, however I may ask you to show how (9) $\Rightarrow$ (10), that'd be a reasonable ?.

§4.6 #11 $y'' + y = \tan^2 t$. Here $y_1 = \cos t$, $y_2 = \sin t$ and $g(t) = \tan^2 t$, $a = 1$.

$$V_1 = \int \frac{-\tan^2 t \sin t}{\cos t \cos t - (-\sin t)\sin t} = \int -\tan^2 t \sin t\, dt$$
$$= \int \frac{-\sin^3 t\, dt}{\cos^2 t}$$
$$= \int \frac{(1-\cos^2 t)}{\cos^2 t}(-\sin t\, dt)$$
$$= \int \frac{1-u^2}{u^2}\, du$$
$$= \frac{-1}{u} - u = \frac{-1}{\cos t} - \cos t = -\sec t - \cos t = V_1$$

$$V_2 = \int \frac{\tan^2 t \cos t\, dt}{1} = \int \frac{\sin^2 t \cos t\, dt}{\cos^2 t}$$
$$= \int \left(\frac{1-\cos^2 t}{\cos t}\right) dt$$
$$= \int (\sec t - \cos t)\, dt$$
$$= \ln|\sec t + \tan t| - \sin t = V_2$$

$$y = y_h + y_p = c_1 \cos t + c_2 \sin t + \cos t(-\sec t - \cos t) + \sin t(-\sin t + \ln|\sec t + \tan t|)$$
$$= c_1 \cos t + c_2 \sin t - 1 - \cos^2 t - \sin^2 t + \sin t \ln|\sec t + \tan t|$$
$$= \boxed{c_1 \cos t + c_2 \sin t - 2 + \sin t \ln|\sec t + \tan t| = y}$$

__Remark:__ Although in previous problems I have not utilized eq^n (10) I think it's clear that it's good to use eq^n (10) and we will usually begin with it as we did here. You should understand where the intial eq^n's for V_1 and V_2 come from, but I will not expect you to justify them every time.

§4.6 #17 $\frac{1}{2}y'' + 2y = \tan 2t - \frac{1}{2}e^t$, I multiply by 2 to begin $y'' + 4y = 2\tan 2t - e^t$. Now we'll break up the problem into sensible pieces

1.) find y_1 & y_2 the fundamental sol$^{\underline{n}}$'s ($y_1 = \cos 2t$ & $y_2 = \sin 2t$)
2.) find y_{p_1} the particular sol$^{\underline{n}}$ to $y'' + 4y = -e^t$
3.) find y_{p_2} the particular sol$^{\underline{n}}$ to $y'' + 4y = 2\tan 2t$
4.) assemble total sol$^{\underline{n}}$ via superposition principle.

To begin then

2.) Let $y_{p_1} = Ae^t$ then $y_{p_1}' = y_{p_1}'' = y_{p_1} = Ae^t$ substitute to get

$$y_{p_1}'' + 4y_{p_1} = 5y_{p_1} = 5Ae^t = -e^t \Rightarrow 5A = -1 \Rightarrow A = \frac{-1}{5}$$

Thus by undetermined coefficients we find $y_{p_1} = \frac{-1}{5}e^t$

3.) Use variation of parameters to find $Y_{p_2} = V_1 y_1 + V_2 y_2$ for the Deq$^{\underline{n}}$ $y'' + 4y = 2\tan 2t = g(t)$. Go straight to eq$^{\underline{n}}$ (10)

$$V_1 = \int \frac{-2\tan(2t)\sin(2t)\,dt}{\sin 2t(-2\cos 2t) - 2\cos 2t(\cos 2t)}$$
$$= \int \frac{\sin^2(2t)\,dt}{\cos(2t)}$$
$$= \int \frac{1-\cos^2(2t)}{\cos(2t)}\,dt$$
$$= \int [\sec(2t) - \cos(2t)]\,dt = \frac{1}{2}\ln|\sec(2t) + \tan(2t)| - \frac{1}{2}\sin(2t) = V_1$$

$$V_2 = \int \frac{2\tan(2t)\cos(2t)\,dt}{-2}$$
$$= \int -\sin(2t)\,dt$$
$$= \frac{1}{2}\cos(2t) = V_2$$

using what we saw for V_1, same here. btw. note the denominator is the Wronskian! that is comforting as $W[Y_1, Y_2](t) \neq 0$ for L.I. functions Y_1 & Y_2.

Thus $Y_{p_2} = \frac{\cos(2t)}{2}\ln|\sec 2t + \tan 2t| - \frac{1}{2}\cancel{\sin 2t\cos 2t} + \frac{1}{2}\cancel{\sin 2t\cos 2t}$

4.) By the superposition principle (for linear differential eq$^{\underline{n}}$'s, Th$^{\underline{m}}$(3) pg. 187)

$$\boxed{y = C_1\cos 2t + C_2\sin 2t - \frac{1}{5}e^t + \frac{1}{2}\cos(2t)\ln|\sec 2t + \tan 2t|}$$

Remark: If we had not split up the problem and instead had tried to do it in one shot with variation of parameters it would have also worked but would've been messy.

WARNING: #8 & #12 are incorrect in their reasoning given here, why? H (37)

§6.1 #8 Is $\{x^2, x^2-1, 5\}$ a linearly independent set of functions on $(-\infty, \infty)$.

Label as customary $f_1(x) = x^2$, $f_2(x) = x^2-1$, $f_3(x) = 5$

$f_1'(x) = 2x$, $f_2'(x) = 2x$, $f_3'(x) = 0$

$f_1''(x) = 2$, $f_2''(x) = 2$, $f_3''(x) = 0$

$$W[f_1,f_2,f_3](x) = \begin{vmatrix} x^2 & x^2-1 & 5 \\ 2x & 2x & 0 \\ 2 & 2 & 0 \end{vmatrix} = x^2\begin{vmatrix} 2x & 0 \\ 2 & 0 \end{vmatrix} - (x^2-1)\begin{vmatrix} 2x & 0 \\ 2 & 0 \end{vmatrix} + 5\begin{vmatrix} 2x & 2x \\ 2 & 2 \end{vmatrix}$$

$$= x^2(2x\cdot 0 - 0\cdot 2) - (x^2+1)(2x\cdot 0 - 0\cdot 2) + 5(4x-4x)$$

$$= 0$$

Therefore the Wronskian of $\{x^2, x^2-1, 5\}$ is zero $\Rightarrow$ they linearly dependent.

Remark: Using the Wronskian here is a little overkill, after all it's not hard to see that $f_1 - f_2 = 1 \Rightarrow 5f_1 - 5f_2 - 5f_3 = 0$. That equality expresses linear dependence directly.

§6.1 #12 $\{\cos 2x, \cos^2 x, \sin^2 x\} = \{f_1(x), f_2(x), f_3(x)\}$

I'll use $\cos^2(x) = \frac{1}{2}(1+\cos 2x)$, $\sin^2(x) = \frac{1}{2}(1-\sin 2x)$ to find the derivatives $f_2'(x) = -\sin(2x)$, $f_3'(x) = -\cos(2x)$

$$W[f_1,f_2,f_3](x) = \begin{vmatrix} \cos 2x & \cos^2 x & \sin^2 x \\ -2\sin 2x & -\sin 2x & -\cos 2x \\ -4\cos 2x & -2\cos 2x & 2\sin 2x \end{vmatrix}$$

$$= \cos(2x)(-2\sin^2 2x - 2\cos^2 2x) - \cos^2 x(-4\sin^2 2x - 4\cos^2 2x) + \sin^2 x(4\sin 2x\cos 2x - 4\sin 2x\cancel{\cos 2x})$$

$$= -2\cos(2x) + 4\cos^2(x)$$

$$= -2\cancel{\cos(2x)} + 2(1 + \cancel{\cos 2x})$$

$$= 2$$

not quite correct, why?

thus by Th^{m} (3) $\{\cos 2x, \cos^2 x, \sin^2 x\}$ are L.I.

§6.1 #18 $y'''' - y = 0$ has fundamental sol^{n} set $\{e^x, e^{-x}, \cos(x), \sin(x)\}$

This terminology simply means these are a set of n- LI functions which are each a sol^{n} to a given n^{th} order DEq^{n}, here $n=4$. You can derive these from $\lambda^4 - 1 = 0 \Rightarrow \lambda^4 = 1$ has sol^{n} $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_{3,4} = \pm i$ which then lead to the functions e^x, e^{-x}, $\cos(x)$ and $\sin(x)$. Let's prove L.I.

$$W = \begin{vmatrix} e^x & e^{-x} & \cos x & \sin x \\ e^x & -e^{-x} & -\sin x & \cos(x) \\ e^x & e^{x} & -\cos x & -\sin x \\ e^x & -e^{x} & \sin x & -\cos x \end{vmatrix} = \underbrace{e^x\begin{vmatrix} -e^{-x} & -\sin x & \cos x \\ e^{-x} & -\cos x & -\sin x \\ -e^{-x} & \sin x & -\cos x \end{vmatrix} - e^{-x}\begin{vmatrix} e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \\ e^x & \sin x & -\cos x \end{vmatrix}}$$

§6.1#18 continued

$$W = 808 + \cos(x)\begin{vmatrix} e^x & -e^{-x} & \cos x \\ e^x & e^{-x} & -\sin x \\ e^x & -e^{-x} & -\cos x \end{vmatrix} - \sin x \begin{vmatrix} e^x & -e^{-x} & -\sin x \\ e^x & e^{-x} & -\cos x \\ e^x & -e^{-x} & \sin x \end{vmatrix}$$

$$= e^x\Big\{-e^{-x}(\cos^2 x+\sin^2 x) + \sin(x)(-e^{-x}\cancel{\cos x} + e^{-x}\sin x) + \cos x\,(e^{-x}\cancel{\sin x} - e^{-x}\cos x)\Big\}$$

$$-e^{-x}\Big\{e^x(\cos^2 x+\sin^2 x) + \sin x(-e^x\cancel{\cos x} + e^x \sin x) + \cos x\,(e^x\cancel{\sin x} + e^x\cos x)\Big\}$$

$$+\cos x\Big\{e^x(-e^{-x}\cos x + e^{-x}\sin x) + e^{-x}(-e^x\cos x + e^x\sin x) + \cos x\,(-e^x e^{-x} - e^{-x}e^x)\Big\}$$

$$-\sin x\Big\{e^x(e^{-x}\sin x - \cos x\, e^{-x}) + e^{-x}(e^x\sin x + e^x\cos x) - \cos x\,(-e^x e^{-x} - e^{-x}e^x)\Big\}$$

$$= e^x\Big\{-e^{-x} - e^{-x}(\sin^2 x + \cos^2 x)\Big\} - e^{-x}\Big\{e^x + e^x(\sin^2 x+\cos^2 x)\Big\}$$

$$+\cos x\Big\{-\cos x + \sin x - \cos x + \sin x - 2\cos x\Big\}$$

$$-\sin x\Big\{\sin x - \cancel{\cos x} + \sin x + \cancel{\cos x} + 2\cos x\Big\}$$

$$= -2 - 2 + \cos x\,(-4\cos x + 2\sin x) + \sin x\,(2\sin x + 2\cos x)$$

$$= -4 - 4\cos^2 x + \cancel{2\sin x\cos x} - 2\sin^2 x - \cancel{2\sin x\cos x}$$

$$= \ldots$$

So something is wrong somewhere, the answer is $W = -8$. I know this as I checked it with my handy TI-89. If you can find my error I'll give you a bonus point, good luck.

- The point here is simply that L.I. can be checked with the Wronskian (although this is special to LI of functions, if we asked about L.I. of vectors in $\mathbb{R}^3$ it'd be difficult to set up the Wronskian, how do you differentiate a vector?)

- Oh, on second thought, the real point is that Linear Independence is an important and central idea to understanding why & what we do throughout most of the semester. The Wronshian just helps us find an explicit method of checking linear independence, (at the price of cumbersome calculation.)

§6.2 #1 $y''' + 2y'' - 8y' = 0$

Characteristic Eq^{n} $\lambda^3 + 2\lambda^2 - 8\lambda = \lambda(\lambda^2 + 2\lambda - 8) = \lambda(\lambda+4)(\lambda-2) = 0$

Third order in λ gives 3-sol^{n}'s $\lambda_1 = 0,\ \lambda_2 = -4,\ \lambda_3 = 2$

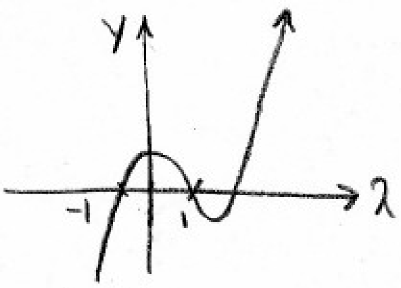
Which then gives 3-fund. sol^{n}'s $y_1 = 1,\ y_2 = e^{-4x},\ y_3 = e^{2x}$

Which then assembles the general sol^{n} $\boxed{y = c_1 + c_2 e^{-4x} + c_3 e^{2x}}$

§6.2#2 $y''' - 3y'' - y' + 3y = 0$

Char. Eq^{n}: $\lambda^3 - 3\lambda^2 - \lambda + 3 = 0$ (How to factor a cubic ! ?)

For most textbook problems there is always some silly natural number root, but looking towards non-textbook problems we can use the following scheme, graph it, find a root then either find the rest graphically or factor out the linear factor corresponding to the root.



$y = \lambda^3 - 3\lambda^2 - \lambda + 3 = f(\lambda)$

apparently $\lambda = 1$ is a zero $\Rightarrow f(\lambda)$ has a $(\lambda - 1)$ factor,

$$\lambda^3 - 3\lambda^2 - \lambda + 3 = (\lambda - 1)(\lambda^2 + b\lambda + c)$$
$$= \lambda^3 + \lambda^2(b-1) + \lambda(c-b) - c$$

Can see by comparing that $b = -2$ and $c = -3$.

Hence $\lambda^3 - 3\lambda^2 - \lambda + 3 = (\lambda - 1)(\lambda^2 - 2\lambda - 3)$

The more algebraically adept among you might just see how to factor this cubic, but my method here allows for messier quadratics and also some quadratics do not have roots so the pure graphical method fails. Pragmatically, we could also use the root finder or polysolve option on most graphical calculators.

- In general factoring polynomials of order greater than 2 is quite a challange. This one is relatively harmless (rational roots !)

$$\lambda^2 - 3\lambda^2 - \lambda + 3 = (\lambda - 1)(\lambda^2 - 2\lambda - 3) = (\lambda - 1)(\lambda - 3)(\lambda + 1) = 0$$

Thus $\lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = -1$ hence

$$\boxed{y = c_1 e^{x} + c_2 e^{3x} + c_3 e^{-x}}$$

§6.2#6 $y'''' - y'' + 2y = 0$

$$\lambda^4 - \lambda^2 + 2 = 0$$

Hint, see #36 of §4.4. You need to do the same trick here, change the quartic eqn to a quadratic eqn. Should find 4 roots.

§6.2#14 $y'''' + 2y''' + 10y'' + 18y' + 9y = 0$

$$\lambda^4 + 2\lambda^3 + 10\lambda^2 + 18\lambda + 9 = 0$$

Hint: $y = \sin(3x)$ is a soln. This means there is at least two imaginary roots namely $\lambda = \pm 3i$. That means we can factor out (λ^2+9) corresponding to the zeros $\pm 3i$. The whole thing is 4th order so if we factor out a quadratic then the rest of the thing must be a quadratic but I'll use algebra to figure out which quadratic it must be. Since λ^4 has a 1 coefficient we need not worry about "a" in $a\lambda^2 + b\lambda + c$, b & c will suffice,

$$\lambda^4 + 2\lambda^3 + 10\lambda^2 + 18\lambda + 9 = (\lambda^2+9)(\lambda^2 + b\lambda + c)$$

$$= \lambda^4 + \lambda^3(b) + \lambda^2(c+9) + \lambda 9b + 9c$$

gotta find b & c to match them up.

Gives overdetermined system of linear eqn's for b & c

$2 = b \Rightarrow b=2$
$10 = c+9 \Rightarrow c=1$
$18 = 9b \Rightarrow b=2$
$9 = 9c \Rightarrow c=1$

it's ok to have extra eqn's so long as they are consistent, these are.

We find then $\lambda^4 + 2\lambda^3 + 10\lambda^2 + 18\lambda + 9 = (\lambda^2+9)(\lambda^2 + 2\lambda + 1)$

Then $\lambda^4 + 2\lambda^3 + 10\lambda^2 + 18\lambda + 9 = (\lambda^2+9)(\lambda+1)^2$ yielding zeros

$$\lambda_{1,2} = \pm 3i, \quad \lambda_3 = -1 = \lambda_4$$

Hence

$$\boxed{y = c_1 \cos 3x + c_2 \sin 3x + c_3 e^{-x} + c_4 x e^{-x}}$$

Remark: this an other pretty textbook polynomials can be factored with the so-called "rational roots" theorem. That'd find alot of the roots in this section. My goal is to make you think a little about how to tackle uglier ones.

§6.2#15 Here D denotes the operator $\frac{d}{dx}$ on $Y(x)$.

$$\underbrace{(D-1)^2(D+3)(D^2+2D+5)^2}_{7^{th}\text{ order polynomial in the operator } D} [Y] = 0 \qquad (*)$$

7^{th} order polynomial in the <u>operator</u> D, multiplication is <u>composition of operators</u>.

$$(D-1)^2[Y] = 0 \quad \Rightarrow \quad Y_1 = e^x \text{ or } Y_2 = xe^x$$

This is straight forward to verify,

$$(D-1)[Y_1](x) = \left(\tfrac{d}{dx} - 1\right)e^x = e^x - e^x = 0.$$

$$(D-1)[Y_2](x) = \left(\tfrac{d}{dx} - 1\right)xe^x = e^x + xe^x - xe^x = e^x$$

$$(D-1)^2[Y_2] = \Big(\underset{\substack{\uparrow \\ \text{Composition}}}{(D-1)\circ(D-1)}\Big)[Y_2] = (D-1)(D-1)[Y_2]$$
$$= (D-1)[Y_1]$$
$$= 0$$

- It's important to intepret $(D-1)(D-1)$ as composition of operators. Anyway using similar logic and noting

$$\lambda^2 + 2\lambda + 5 = 0 \Rightarrow \lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i \quad \text{implies},$$

$$(D^2+2D+5)^2[Y] = 0 \quad \text{has} \quad \left.\begin{array}{l} Y_3 = e^{-x}\cos 2x \\ Y_4 = e^{-x}\sin 2x \end{array}\right\} \text{killed by } [D^2+2D+5]$$
$$\left.\begin{array}{l} Y_5 = xe^{-x}\cos 2x \\ Y_6 = xe^{-x}\sin 2x \end{array}\right\} \text{killed by } [D^2+2D+5]^2$$

And of course $(D+3)[Y] = 0$ has $Y_7 = e^{-3x}$ as a sol$^{\underline{n}}$.
In total we have an eq$^{\underline{n}}$ which is a linear $7^{\underline{th}}$ order ODE, notice if Y_i is a sol$^{\underline{n}}$ to any of the factors it's a sol$^{\underline{n}}$ to the whole composition of operators, just one them needs to kill it. So $Y_1, Y_2, \dots, Y_7$ are all sol$^{\underline{n}}$'s to $(*)$

$$\boxed{Y = c_1e^x + c_2xe^x + c_3e^{-x}\cos 2x + c_4e^{-x}\sin 2x + c_5e^{-x}x\cos 2x + c_6xe^{-x}\sin 2x + c_7e^{-3x}}$$

<u>Remark:</u> A linear ODE of the form $L[Y] = 0$ where L is a polynomial $P(D)$ where $D = \frac{d}{dx}$ has char. eq$^{\underline{n}}$ $P(\lambda)$, same algebra!

§6.3 # 9 $y''' - 3y'' + 3y' - y = e^x \quad (*)$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 = 0$$

Thus the associated homog. eq^n has form $(D-1)^3[y] = 0$. Note e^x is anihilated by $(D-1)$. Thus every sol^n of $(*)$ must satisfy

$$(D-1)^3(D-1)[y] = (D-1)^4[y] = 0 \quad (**)$$

The general sol^n to $(**)$ is $\left((\lambda-1)^4 = 0 \Rightarrow \lambda_1 = 1 = \lambda_2 = \lambda_3 = \lambda_4\right)$

$$y = \underbrace{c_1 e^x + c_2 x e^x + c_3 x^2 e^x}_{y_h} + \underbrace{c_4 x^3 e^x}_{y_p}$$

- This method is neat in that it translates the problem into a corresponding homogeneous problem. We already knew to guess $y_p = Ax^3 e^x$ since we'd have observed to begin that $y_h = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$ thus the naive guess $y_p(\text{naive}) = Ae^x$ overlaps with y_1 then $y_p(\text{less naive}) = xAe^x$ overlaps with y_2 then $y_p(\text{even less naive}) = x^2 A e^x$ still overlaps with $y_3 = x^2 e^x$, finally we'd have "guessed" that $y_p = Ax^3 e^x$. Notice how the Anihilator Method has discovered the x^3 through alternate reasoning. We view the anihilator method as one proof of our previous method, of course the fact it works is a proof in itself.

$$y_p = Ax^3 e^x$$
$$y_p' = A(3x^2 + x^3)e^x$$
$$y_p'' = A(6x + 6x^2 + x^3)e^x$$
$$y_p''' = A(6 + 18x + 9x^2 + x^3)e^x$$

$$y_p''' - 3y_p'' + 3y_p' - y_p = e^x$$

$$A(6 + 18x + 9x^2 + x^3)e^x + -3A(6x + 6x^2 + x^3)e^x + +3A(3x^2 + x^3)e^x + -Ax^3 e^x = e^x$$

$$x^3\underbrace{(A - 3A + 3A - A)}_{0} + x^2\underbrace{(9A - 18A + 9A)}_{0} + x\underbrace{(18A - 18A)}_{0} + 6A = 1$$

$$\Rightarrow A = 1/6$$

Thus $\boxed{y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + \frac{1}{6} x^3 e^x}$

§6.3 #12 find an operator A which anihilates $f(x) = 3x^2 - 6x + 1$.

Easy use D^3, $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}(3x^2 - 6x + 1)\right)\right) = 0 \therefore A[f] = 0$.

§6.3 #14 try $A = D - 5$.

§6.3 #17 $\underbrace{x^2 e^{-x} \sin 2x}$

sol^n from $\lambda = -1 \pm 2i$
which has corresponding $(\lambda + 1 - 2i)(\lambda + 1 + 2i) = \lambda^2 + 2\lambda + 5$ char eq$^{\underline{n}}$.
which has corresponding operator $D^2 + 2D + 5$ that is,

$$(D^2 + 2D + 5)[e^{-x} \sin 2x] = 0.$$

The x^2 suggests this is the sol^n to a D Eq$^{\underline{n}}$ with not just one $(\lambda^2 + 2\lambda + 5)$ factor in the characteristic eq$^{\underline{n}}$. To get upto $x^2 e^{-x} \sin 2x$ we'd need to have already

$$\underbrace{\underbrace{\underbrace{e^{-x}(\sin 2x), \ e^{-x}\cos 2x}_{(D^2+2D+5)[Y]=0}, \ xe^{-x}\sin 2x, \ xe^{-x}\cos 2x}_{(D^2+2D+5)^2[Y]=0}, \ x^2e^{-x}\sin 2x, \ x^2e^{-x}\cos 2x}_{(D^2+2D+5)^3[Y]=0}$$

Identify that $x^2 e^{-x} \sin 2x$ is one of the fundamental sol^n's to $(D^2 + D + 5)^3 [Y] = 0$. So clearly $\boxed{A = (D^2 + 2D + 5)^3}$

Remark: I'm just trying to validate the books list on pg. 334. You don't need to do all the writing I'm doing here. You may just remember (i) $\longrightarrow$ (iv) of p. 334 to help find Anihilators for suitable $f(x)$.

§6.4 #2 $y''' - 2y'' + y' = x$ (only for practice, pragmatically speaking undetermined coeff. would be easier)

Char. Eq^{n} $\lambda^3 - 2\lambda^2 + \lambda = 0$

$$\lambda(\lambda^2 - 2\lambda + 1) = \lambda(\lambda-1)^2 = 0 \quad \therefore \lambda_1 = 0, \ \lambda_2 = 1 = \lambda_3$$

Hence, fund. sol^{n}'s are $y_1 = 1$, $y_2 = e^x$, $y_3 = xe^x$. We will suppose the particular sol^{n} exists and has form $y_p = V_1 y_1 + V_2 y_2 + V_3 y_3$. Our job is then to find coefficient functions V_1, V_2 and V_3. We'll need the Wronskian so lets get it done,

$$W = \begin{vmatrix} 1 & e^x & xe^x \\ 0 & e^x & (1+x)e^x \\ 0 & e^x & (2+x)e^x \end{vmatrix} = 1(e^x(2+\cancel{x})e^x - (1+\cancel{x})e^x e^x = e^{2x}$$

Now we'll use eq^{n} (11), note $g(x) = x$, and W_k is described in eq^{n}(10).

$$\boxed{V_1 = \int \frac{x\,W_1(x)}{e^{2x}}dx \quad \Big| \quad V_2 = \int \frac{x\,W_2(x)}{e^{2x}}dx \quad \Big| \quad V_3 = \int \frac{x\,W_3(x)}{e^{2x}}dx} \quad (\star)$$

Ok, so we should calculate $W_i(x)$ before, oops.

$$W_1(x) = (-1)^{3-1} W[e^x, xe^x] = \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)e^x \end{vmatrix} = e^{2x}(1+x-x) = e^{2x}.$$

$$W_2(x) = (-1)^{3-2} W[1, xe^x] = -\begin{vmatrix} 1 & xe^x \\ 0 & (1+x)e^x \end{vmatrix} = (1+x)e^x(-1)$$

$$W_3(x) = (-1)^{3-3} W[1, e^x] = \begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix} = e^x$$

Hence, returning to $(\star)$

$$V_1 = \int x\,dx = \tfrac{1}{2}x^2$$

$$V_2 = -\int \frac{x(1+x)e^x}{e^{2x}}dx = -\int e^{-x}(x+x^2)dx = (-x^2-3x-3)e^{-x}(-1)$$

$$V_3 = \int \frac{xe^x}{e^{2x}}dx = \int e^{-x}x\,dx = (-x-1)e^{-x}$$

Thus

$$y_p = \tfrac{1}{2}x^2 + (x^2+3x+3)e^{-x}e^x - (x+1)e^{-x}xe^x$$

$$= \tfrac{1}{2}x^2 + \cancel{x^2} + 3x + 3 - \cancel{x^2} - x$$

$$= \boxed{\tfrac{-1}{2}x^2 + 2x + 3 = y_p}$$

not hard to check

$y_p'' = 1$

$y_p' = x + 2$

$-2y_p'' + y_p' = -2 + x + 2 = x$