§7.2 #1 Let $f(t) = t$ calculate the Laplace transform of f; $\mathcal{L}\{f\}(s)$,

$$\mathcal{L}\{f\}(s) \equiv F(s) = \int_0^\infty e^{-st} t\, dt$$

$$= \lim_{N\to\infty} \int_0^N e^{-st} t\, dt$$

So calculate the integral and then finish it,

$$\int \underset{u}{\underset{\sim}{t}}\, \underbrace{e^{-st}dt}_{dV} = \underbrace{-\frac{1}{s}te^{-st} - \int -\frac{1}{s}e^{-st}dt}_{\text{Int. by Parts.}} = \frac{1}{s}\left(-te^{-st} - \frac{1}{s}e^{-st}\right)$$

So returning to the Laplace transform,

$$F(s) = \lim_{N\to\infty}\left(\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2}\right)\Bigg|_{0=t}^{N=t}$$

$$= \lim_{N\to\infty}\left(\underbrace{-\frac{Ne^{-sN}}{s}}_{\to 0} - \underbrace{\frac{e^{-sN}}{s^2}}_{\to 0} + \frac{1}{s^2}\right) = \boxed{\frac{1}{s^2}}$$

assuming $s>0$

We noticed $\lim_{N\to\infty}(Ne^{-sN}) = \lim_{N\to\infty}\left(\frac{N}{e^{sN}}\right) \overset{L}{=} \lim_{N\to\infty}\left(\frac{1}{se^{sN}}\right) = 0$ to see why the $-Ne^{-sN}/s$ term vanished.

§7.2 #4 find Laplace transform of $f(t) = te^{3t}$

$$F(s) = \lim_{N\to\infty}\int_0^N e^{-st}te^{3t}\,dt$$

$$= \lim_{N\to\infty}\int_0^N te^{t(3-s)}\,dt$$

$$= \lim_{N\to\infty}\left(\frac{1}{3-s}te^{t(3-s)}\Big|_0^N - \int_0^N \frac{1}{3-s}e^{t(3-s)}\,dt\right)$$

I.B.P
$u = t$
$dv = e^{t(3-s)}dt$

$$= \lim_{N\to\infty}\left(\cancel{\frac{N}{3-s}e^{N(3-s)}}^{\,0} - \frac{1}{(3-s)^2}\left[\cancel{e^{N(3-s)}}^{\,0} - 1\right]\right)$$

$$= \frac{1}{(3-s)^2} = \boxed{\frac{1}{(s-3)^2} = \mathcal{L}\{te^{3t}\}(s)}$$

assuming $s>3$ to insure the integral converges.

§7.2 #11

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^\pi \sin(t) e^{-st}\, dt + \cancel{\int_\pi^\infty 0 \cdot e^{-st}\, dt} \to 0$$

(*)

$$= \left(\frac{-\cos t}{s^2+1} - \frac{s \sin t}{s^2+1} \right) e^{-st} \Big|_{0=t}^{\pi=t}$$

$$= \frac{-\cos \pi}{s^2+1} e^{-s\pi} + \frac{\cos(0)}{s^2+1}$$

$$= \boxed{\frac{e^{-s\pi}}{s^2+1} + \frac{1}{s^2+1}}$$

(no restriction on s here)

The integral used at (*) is derived by IBP,

$$I = \int \underbrace{\sin t}_{u}\, \underbrace{e^{at}\, dt}_{dV} = \frac{1}{a} \sin t\, e^{at} - \int \frac{1}{a} e^{at} \cos t\, dt$$

$$= \frac{1}{a} e^{at} \sin t - \frac{1}{a}\left(\frac{1}{a} \cos t\, e^{at} + \int \frac{1}{a} e^{at} \sin t\, dt \right)$$

$$= e^{at}\left(\frac{1}{a} \sin t - \frac{1}{a^2} \cos t \right) - \frac{1}{a^2} \underbrace{\int e^{at} \sin t\, dt}_{I}$$

$I \leftarrow$ full circle so solve for I.

$$I\left(1 + \frac{1}{a^2}\right) = e^{at}\left(\frac{\sin t}{a} - \frac{\cos t}{a^2} \right)$$

$$\Rightarrow \quad I = e^{at} \frac{1}{1 + 1/a^2} \left(\frac{\sin t}{a} - \frac{\cos t}{a^2} \right) = e^{at}\left(\frac{a \sin t}{a^2+1} - \frac{\cos t}{a^2+1} \right)$$

§7.2 #17

$$\mathcal{L}\{ e^{3t} \sin 6t - t^3 + e^t \} = \mathcal{L}\{e^{3t} \sin 6t\} - \mathcal{L}\{t^3\} + \mathcal{L}\{e^t\}$$

$$= \boxed{\frac{6}{(s-3)^2 + 36} - \frac{6}{s^4} + \frac{1}{s-1}}$$

using Table on p. 358

§7.2 #18

$$\mathcal{L}\{ t^4 - t^2 - t + \sin \sqrt{2} t \} = \mathcal{L}\{t^4\} - \mathcal{L}\{t^2\} - \mathcal{L}\{t\} + \mathcal{L}\{\sin \sqrt{2} t\}$$

$$= \boxed{\frac{24}{s^5} - \frac{2}{s^3} - \frac{1}{s^2} + \frac{\sqrt{2}}{s^2 + 2}}$$

§7.3 #5 Hard part first,

$$\mathcal{L}\{2t^2e^{-t}\} = 2\,\mathcal{L}\{t^2e^{-t}\} \underset{Th^m(3)}{=} 2F(s+1) = 2\frac{2!}{(s+1)^3} = \frac{4}{(s+1)^3}$$

Now use what we just learned, plus table 7.1 to get the next 2 terms,

$$\mathcal{L}\{2t^2e^{-t} - t + \cos 4t\} = \boxed{\frac{4}{(s+1)^3} - \frac{1}{s^2} + \frac{s}{s^2+16}}$$

§7.3 #6 Use Th^m(3) that says $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ where $\mathcal{L}\{f\}(s) = F(s)$.

$$\mathcal{L}\{e^{3t}t^2\}(s) = F_1(s-3) = \frac{2}{(s-3)}$$

$$\mathcal{L}\{e^{-2t}\sin 2t\}(s) = F_2(s+2) = \frac{2}{(s+2)^2+4}$$

(Used $\mathcal{L}\{t^2\}(s) = 2/s^3$; used $\mathcal{L}\{\sin 2t\}(s) = \frac{2}{s^2+4}$)

Thus $\mathcal{L}\{e^{3t}t^2 + e^{-2t}\sin 2t\} = \boxed{\frac{2}{s-3} + \frac{2}{(s+2)^2+4}}$

§7.3 #12 $\sin 3t \cos 3t$ can be converted into a sum of sines and cosines by the right trig identities, or use the complex exponentials $\sin(3t) = \frac{1}{2i}(e^{3it} - e^{-3it})$ and $\cos 3t = \frac{1}{2}(e^{3it} + e^{-3it})$

Just a particular application of $\sin\theta = \frac{1}{2}(e^{i\theta} - e^{-i\theta})$ and $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$. These followed naturally from $e^{i\theta} = \cos\theta + i\sin\theta$.

Anyway lets get to work,

$$\begin{aligned}\sin(3t)\cos(3t) &= \frac{1}{2i}(e^{3it} - e^{-3it})\frac{1}{2}(e^{3it} + e^{-3it}) \\ &= \frac{1}{4i}(e^{6it} + 1 - 1 - e^{-6it}) \\ &= \frac{1}{2}\frac{1}{2i}(e^{6it} - e^{-6it}) \\ &= \frac{1}{2}\sin(6t)\end{aligned}$$

we derived $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$ for $\theta = 3t$

Now it's easy $\mathcal{L}\{\sin(3t)\cos(3t)\}(s) = \mathcal{L}\{\frac{1}{2}\sin 6t\}(s) = \frac{1}{2}\frac{6}{s^2+36} = \boxed{\frac{3}{s^2+36}}$

§7.3 #15

$$\begin{aligned}\cos^3 t &= \frac{1}{8}(e^{it}+e^{-it})(e^{it}+e^{-it})(e^{it}+e^{-it}) \\ &= \frac{1}{8}(e^{2it} + 2 + e^{-2it})(e^{it}+e^{-it}) \\ &= \frac{1}{8}(e^{3it} + e^{it} + 2e^{it} + 2e^{-it} + e^{-it} + e^{-3it}) \\ &= \frac{1}{4}\frac{1}{2}(e^{3it} + e^{-3it}) + \frac{3}{4}\frac{1}{2}(e^{it} + e^{-it}) \\ &= \frac{1}{4}\cos(3t) + \frac{3}{4}\cos t.\end{aligned}$$

(Check it with TI-89 tCollect(cos(x)^3))

Then $\mathcal{L}\{\cos^3 t\} = \boxed{\frac{1}{4}\frac{s}{s^2+9} + \frac{3}{4}\frac{s}{s^2+1}}$

§7.3 #22 Starting with $\mathcal{L}\{1\}(s) = \frac{1}{s}$ use the theorem
$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{ds^n}(s)$ to derive the f-la $\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$
We take $f(t) = 1$ in the theorem then calculate

$$\mathcal{L}\{t^n\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s}\right) = (-1)^n \frac{d^{n-1}}{ds^{n-1}}\left(\frac{-1}{s^2}\right)$$

$$= (-1)^n \frac{d^{n-2}}{ds^{n-2}}\left(\frac{(-1)(-2)}{s^3}\right)$$

$$= (-1)^n (-1)^2 \frac{d^{n-2}}{ds^{n-2}}\left(\frac{2\cdot 1}{s^3}\right)$$

$$= (-1)^n (-1)^3 \frac{d^{n-3}}{ds^{n-3}}\left(\frac{3\cdot 2\cdot 1}{s^4}\right)$$

$$= (-1)^n (-1)^n \frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{s^{n+1}}$$

$$= \boxed{\frac{n!}{s^{n+1}} = \mathcal{L}\{t^n\}(s)}$$

§7.4#1 $G(s) = \frac{6}{(s-1)^4}$ looks like $\frac{3!}{s^2}$ but it's shifted by 1. So we should think about Th^m (3) of p. 260. $\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a)$
Let $F(s) = 3!/s^4$ and $a = 1$ then recall from table 7.1 that $\mathcal{L}\{t^3\} = \frac{3!}{s^4}$
put it all together,

$$\mathcal{L}\{e^t t^3\}(s) = \frac{3!}{(s-1)^4} = \frac{6}{(s-1)^4}$$

$$\therefore \boxed{\mathcal{L}^{-1}\left\{\frac{6}{(s-1)^4}\right\}(t) = e^t t^3 = g(t) = \mathcal{L}^{-1}\{G\}(t)}$$

Remark: In terms of calculational difficulty taking the Laplace transform of f to get F is analogous to differentiation, in the sense that it is straight-forward, you just use the table & properties of the Laplace transform. On the other hand, taking the inverse Laplace transform is more like integration, you have to see where your going to get there, it's much harder. To take the inverse transform basically we just have to make an informed guess and then check our guess.

§7.4#2 $F(s) = \frac{2}{s^2+4}$ notice $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$

Thus $\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}(t) = \boxed{\sin(2t) = f(t)}$

§7.4#3 $F(s) = \frac{s+1}{s^2+2s+10}$. Since $s^2+2s+10$ cannot be factored over the real numbers $\Rightarrow$ we'll get sines & cosines.

Complete the square: $s^2+2s+10 = (s+1)^2+9$. Thus

$$\frac{s+1}{s^2+2s+10} = \frac{s+1}{(s+1)^2+9} = \mathcal{L}\{e^{-t}\cos(3t)\}(s) \quad \left(\text{last entry of table.}\right)$$

$$\therefore \boxed{\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\}(t) = e^{-t}\cos(3t)}$$

§7.4 # 8 $F(s) = \frac{1}{s^5} = \frac{1}{4!}\frac{4!}{s^5} = \frac{1}{4!}\mathcal{L}\{t^4\}(s) = \mathcal{L}\{\frac{t^4}{24}\}(s)$

$$\Rightarrow \boxed{\mathcal{L}^{-1}\{1/s^5\}(t) = t^4/24}$$

§7.4 # 9 $\frac{3s-15}{2s^2-4s+10} = F(s)$. We need to refine the expression algebraically so we can see what to do. Notice,

$$2s^2-4s+10 = 0 \quad \text{when} \quad s = \frac{4 \pm \sqrt{16-80}}{4} = 1 \pm \frac{8i}{4}$$

So it cannot be factored over $\mathbb{R}$, so complete the square,

$$2s^2-4s+10 = 2(s^2-2s+5) = 2((s-1)^2+4)$$

I like to give the s^2 term a coefficient of <u>one</u> before completing square. Just makes it easier.

Thus the denominator suggests we'll get $\sin(2t)$ or $\cos(2t)$ apparently shifted by e^t. Let's work out the details,

$$\frac{3s-15}{2s^2-4s+10} = \frac{1}{2}\left(\frac{3s-15}{(s-1)^2+4}\right)$$
$$= \frac{1}{2}\left(\frac{3(s-1)+3-15}{(s-1)^2+4}\right)$$
$$= \frac{3}{2}\left(\frac{s-1}{(s-1)^2+4}\right) - \frac{6}{(s-1)^2+4}$$
$$= \frac{3}{2}\left(\frac{s-1}{(s-1)^2+2^2}\right) - 3\left(\frac{2}{(s-1)^2+2^2}\right)$$
$$= \frac{3}{2}\mathcal{L}\{e^t \sin 2t\} - 3\mathcal{L}\{e^t \cos 2t\}$$

I want a function of $(s-1)$ so I add & subtract to make that happen explicitly.

trying to manipulate it till it looks like last entries in table 7.1

Thus in view of the algebra above

$$\boxed{\mathcal{L}^{-1}\left\{\frac{3s-15}{2s^2-4s+10}\right\}(t) = \frac{3}{2}e^t \sin 2t - 3e^t \cos(2t)}$$

<u>Remark</u>: to begin you must determine the nature of the quadratic in the denominator. If it's irreducible (can't factor) then complete the square and proceed as I did above. Or you can be boring and use logic like example 7 of p. 373.

§7.4 #21 $\dfrac{6s^2-13s+2}{s(s-1)(s-6)} = \dfrac{A}{s} + \dfrac{B}{s-1} + \dfrac{C}{s-6}$ ← partial fractions aka. undoing a common-denominator

$$6s^2-13s+2 = A(s-1)(s-6)+Bs(s-6)+Cs(s-1)$$

$s=0$ | $2 = A(-1)(-6) = 6A \Rightarrow A = 1/3$

$s=1$ | $6-13+2 = -5 = -5B \Rightarrow B = 1$

$s=6$ | $216-78+2 = 140 = C\cdot 30 \Rightarrow C = 14/3$

$$\Rightarrow \quad \frac{6s^2-13s+2}{s(s-1)(s-6)} = \frac{1}{3}\left(\frac{1}{s}\right) + \frac{1}{s-1} + \frac{14}{3}\left(\frac{1}{s-6}\right)$$

Thus $\boxed{\mathcal{L}^{-1}\left\{\frac{6s^2-13s+2}{s(s-1)(s-6)}\right\}(t) = \frac{1}{3}t + e^t t + \frac{14}{3}te^{6t}}$

§7.4 #22 $\dfrac{s+11}{(s-1)(s+3)} = \dfrac{A}{s-1} + \dfrac{B}{s+3}$

$s+11 = A(s+3) + B(s-1) \xrightarrow{s=-3} 8 = -4B \therefore B = -2$

$s+11 = A(s+3) + B(s-1) \xrightarrow[s=1]{} 12 = 4A \therefore A = 3$

Thus $\dfrac{s+11}{(s-1)(s+3)} = \dfrac{3}{s-1} - \dfrac{2}{s+3} = F(s)$

$$\Rightarrow \boxed{\mathcal{L}^{-1}\{F\}(t) = 3e^t - 2e^{-3t}}$$

§7.4 # 26 $\dfrac{7s^3-2s^2-3s+6}{s^3(s-2)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s^3} + \dfrac{D}{s-2}$

$$\begin{aligned} 7s^3-2s^2-3s+6 &= As^2(s-2) + Bs(s-2) + C(s-2) + Ds^3 \\ &= s^3[A+D] + s^2[-2A+B] + s[-2B+C] - 2C \end{aligned}$$

Equating Coefficients gives us,

s^3: $A+D = 7$

s^2: $-2A+B = -2$

s: $-2B+C = -3$

1: $-2C = 6$

logic here works ↑ the equations starting with the last one,

$\rightarrow C = -3 \rightarrow -2B - 3 = -3 \Rightarrow B = 0 \Rightarrow A = 1$
$\Rightarrow D = 6$

$$\mathcal{L}^{-1}\{F\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{3}{s^3} + \frac{6}{s-2}\right\}(t) = \boxed{1 - \frac{3}{2}t^2 + 6e^{2t}}$$

§7.4 #28 We'll solve for $F(s)$,

$$s^2F(s) + sF(s) - 6F(s) = \frac{s^2+4}{s^2+s}$$

$$F(s)\,(s^2+s-6) = \frac{s^2+4}{s^2+s} \Rightarrow F(s) = \frac{s^2+4}{(s^2+s-6)(s^2+s)}$$

We can factor $(s^2+s-6)(s^2+s) = (s+3)(s-2)s(s+1)$. Then do partial fractions to find (after some algebra)

$$F(s) = \frac{s^2+4}{(s^2+s-6)(s^2+s)} = \frac{-13}{30(s+3)} + \frac{5}{6(s+1)} + \frac{4}{15(s-2)} - \frac{2}{3s}$$

Then it's clear how to find the inverse transform,

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = \boxed{\frac{-13}{30}e^{-3t} + \frac{5}{6}e^{-t} + \frac{14}{5}e^{2t} - \frac{2}{3}}$$

§7.4 #31

a.) $f_1(t) = \begin{cases} 0 & t=2 \\ t & t\neq 2 \end{cases}$

$$\mathcal{L}\{f_1\}(s) = \int_0^\infty f_1(t)e^{-st}dt = \int_0^\infty te^{-st}dt = \mathcal{L}\{t\}(s) = \frac{1}{s^2}$$

b.) $f_2(t) = \begin{cases} 5 & t=1 \\ 2 & t=6 \\ t & t\neq 1,6 \end{cases}$ again $\mathcal{L}\{f_2\}(s) = \mathcal{L}(t)(s)$ the integral ignores a finite number of jump-discontinuities.

c.) $\mathcal{L}\{f_3\}(s) = \mathcal{L}\{t\}(s) = 1/s^2$

<u>ALL</u> of the above are "the" inverse Laplace transform of $1/s^2$.

§7.4 # 33 $\mathcal{L}^{-1}\left\{ \frac{d^nF}{ds^n} \right\}(t) = (-t)^n f(t)$ use this to find the $\mathcal{L}^{-1}$ of $\ln((s+2)/(s-5))$. Notice

$$\ln\left(\frac{s+2}{s-5}\right) = \ln(s+2) - \ln(s-5) = F(s)$$

$$\frac{dF}{ds} = \frac{1}{s+2} - \frac{1}{s-5}$$

$$\mathcal{L}^{-1}\left\{\frac{dF}{ds}\right\}(t) = -tf(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+2} - \frac{1}{s-5}\right\}$$

$$= e^{-2t} - e^{5t}$$

$$\therefore \boxed{f(t) = \frac{-1}{t}\left(e^{-2t} - e^{5t}\right)}$$

§7.4 # 36

$F(s) = \tan^{-1}(1/s)$ find $\mathcal{L}^{-1}\{F\}(t) = f(t)$,

$$\frac{dF}{ds} = \frac{1}{1+(1/s)^2}\cdot\frac{-1}{s^2} = \frac{-1}{s^2+1} = -\mathcal{L}\{\sin t\}(s)$$

$$\mathcal{L}^{-1}\left\{\frac{dF}{ds}\right\}(t) = -tf(t) = -\sin(t) \quad \therefore \boxed{f(t) = \frac{\sin t}{t}}$$

§7.5 #1 Solve $y'' - 2y' + 5y = 0$ subject to $y(0) = 2$, $y'(0) = 4$

$$\mathcal{L}\{y''\} = s^2 Y - 2s - 4$$
$$\mathcal{L}\{y'\} = sY - 2$$
} from Table 7.2 using our intial conditions.

Take the Laplace transform of the DEq^n to obtain,

$$s^2 Y - 2s - 4 - 2(sY - 2) + 5Y = 0$$

$$Y(s^2 - 2s + 5) = 2s + 4 - 4$$

$$Y = \frac{2s}{s^2 - 2s + 5} = \frac{2s}{(s-1)^2 + 4} = \frac{2(s-1) + 2}{(s-1)^2 + 4}$$

$$\mathcal{L}^{-1}\{Y\}(t) = \mathcal{L}^{-1}\left\{2\frac{s-1}{(s-1)^2+4} + \frac{2}{(s-1)^2+4}\right\}(t)$$
$$= \boxed{2e^t \cos(2t) + e^t \sin(2t) = y}$$

Remark: A good check on the algebra is that if you do it correctly you ought to see the characteristic eq^n $as^2 + bs + c$ appear as the coefficient of Y. In the above we saw $s^2 - 2s + 5$ appear.

§7.5 #4

$$\mathcal{L}\{y'' + 6y' + 5y\} = \mathcal{L}\{12e^t\} \; ; \; y(0) = -1 \text{ \& } y'(0) = 7$$

$$[s^2 Y + s\,y(0) - y'(0)] + 6[sY - y(0)] + 5Y = \frac{12}{s-1}$$

$$(s^2 + 6s + 5)Y - s - 7 + 6s = \frac{12}{s-1}$$

$$Y = \frac{1}{s^2 + 6s + 5}\left[\frac{12}{s-1} - 5s + 7\right]$$

$$= \frac{1}{(s+5)(s+6)}\left[\frac{12 - 5s(s-1) + 7(s-1)}{s-1}\right] = \frac{A}{s+5} + \frac{B}{s+6} + \frac{C}{s-1}$$

Need to do partial fractions as indicated, need to find A, B \& C,

§7.5#4 Continued Completing partial fractions algebra yields,

$$Y(s) = \frac{12 + (7-5s)(s-1)}{(s-1)(s+5)(s+1)} = \frac{-15}{2(s+5)} + \frac{3}{2(s+1)} + \frac{1}{s-1}$$

Then the inverse transform reveals

$$y(t) = \mathcal{L}^{-1}\{Y\}(t) = \boxed{-\frac{15}{2}e^{-5t} + \frac{3}{2}e^{-t} + e^{t} = y}$$

Remark: we know another method to solve this type of problem. I think the other method is easier because partial fractions is often time consuming to complete correctly. It is still nice to have another way to do problems we can already do with undetermined coefficients.

§7.5 #5 $w'' + w = t^2 + 2$ subject to $w(0) = 1$ and $w'(0) = -1$

$$s^2 W + s + 1 + W = \mathcal{L}\{t^2 + 2\} = \frac{2}{s^3} + \frac{2}{s}$$

$$(s^2+1)W = \frac{2}{s^3} + \frac{2}{s} - s - 1 = \frac{1}{s^3}(2 + 2s^2 - s^4 - s^3)$$

$$W = \frac{2 + 2s^2 - s^3 - s^4}{s^3(s^2+1)} \overset{(*)}{=} \frac{-s}{s^2+1} - \frac{1}{s^2+1} + \frac{2}{s^3}$$

nonobvious step (*) use partial fractions to prove it.

$$\mathcal{L}^{-1}\{W\}(t) = \boxed{w(t) = -\cos(t) - \sin(t) + t^2}$$

Remark: The partial fractions algebra I've omitted here is straightforward but tedious, you should try it, but you should also find a method to check your work.
- I like to use the TI-89's "expand" function. technology should be used to check your work at this stage in the game.

§7.5#10 Solve $Y'' - 4Y = 4t - 8e^{-2t}$ with $Y(0) = 0$, $Y'(0) = 5$

$$s^2 \bar{Y} - 5 - 4\bar{Y} = \frac{4}{s^2} - \frac{8}{s+2}$$

$$(s^2-4)\bar{Y} = \frac{4}{s^2} - \frac{8}{s+2} + 5$$

$$\bar{Y} = \frac{1}{s^2-4}\left(\frac{4}{s^2} - \frac{8}{s+2} + 5\right)$$

making a common denominator would lead to a denom. of $(s-2)(s+2)^2 s^2$ hence the partial fractal decomposition is a good idea.

$$= \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s} + \frac{D}{s^2} + \frac{E}{(s+2)^2}$$

In contrast to #4 or #5 I have not bothered to make the common denominator explicitly. I suspect in retrospect this will save labor, multiply by the denom. to get, noting $(s-2)(s+2) = s^2-4$ and that just cancells from outset,

$$\frac{\cancel{s^2-4}}{\cancel{s^2-4}}\left[\frac{4}{s^2}(s+2)s^2 - \frac{8}{s+2}(s+2)s^2 + 5(s+2)s^2\right] =$$

$$= 4(s+2) - 8s^2 + 5(s+2)s^2$$

$$= (s+2)[4 + 5s^2] - 8s^2$$

$$= A(s+2)^2 s^2 + B(s-2)(s+2)s^2 + C(s-2)(s+2)^2 s + D(s+2)^2(s-2) + E(s-2)s^2$$

Now put in some nice choices for values of s to obtain eq^n's for determining A, B, C, D, E,

$\underline{s=0}$ $\quad 8 = -8D \Rightarrow D = -1$

$\underline{s=2}$ $\quad 4[4+20] - 32 = 64 = 64A \Rightarrow A = 1$

$\underline{s=-2}$ $\quad -32 = -4(4)E \Rightarrow E = 2$

$\underline{s=1}$ $\quad 19 = 9A - 3B - 9C - 9D - E = 9 - 3B - 9C + 9 - 2 = 16 - 3B - 9C = 19$

$\underline{s=3}$ $\quad 5[4+45] - 72 = 173 = 75C + 45B + 218 \Rightarrow 75C + 45B = -45$

$$-3B - 9C = 3$$

$$+\left(3B + \frac{75}{15}C = \frac{-45}{15} = -3\right)$$

$$\left(\frac{75}{15} - 9\right)C = 0 \Rightarrow C = 0 \Rightarrow B = -1$$

§7.5 #10 Continued

We found after some tedious calculation that

$$\mathcal{Y}(s) = \frac{1}{s-2} - \frac{1}{s+2} - \frac{1}{s^2} + \frac{2}{(s+2)^2}$$

Now we can take the inverse Laplace transform with ease,

$$y(t) = e^{2t} - e^{-2t} - t + 2te^{-2t}$$
$$= \boxed{2\sinh(2t) - t + 2te^{-2t} = y}$$

<u>Remark:</u> the partial fractions algebra was not easy here, it seems to me that the repeated factor s^2 was the main source of the difficulty. It is wise to check your work with technology on such problems. However, you should complete a # of these problems to build your skill.

§7.5 #14 $y'' + y = t$ $\quad y(\pi) = 0$ & $y'(\pi) = 0$

$$s^2 Y + Y = \mathcal{L}\{t\}(s) = \frac{1}{s^2}$$

$$Y(s^2+1) = \frac{1}{s^2} \Rightarrow Y = \frac{1}{s^2(s^2+1)}$$

Need to break it up via partial fractions algebra,

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$1 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2 \quad (*)$$

Let's use complex arithmatic for a change. Over $\mathbb{C}$ a polynomial of order n always has n-roots. Here the roots are 0, i and $-i$.

$\underline{s=0)}$ $\boxed{1 = B}$

$\underline{s=i)}$ $1 = (Ci+D)i^2 = -iC - D$ $\quad$ Eq^n (1)

$\underline{s=-i)}$ $1 = (-Ci+D)(-i)^2 = iC - D$ $\quad$ Eq^n (2)

Then Eq^n (1) + Eq^n (2) $\Rightarrow 2 = -2D \Rightarrow \boxed{D=-1}$

Eq^n(1) − Eq^n(2) $\Rightarrow 0 = -2iC \Rightarrow \boxed{C=0}$

Unforunately we don't have enough data to find A yet (the repeated root zero is the source of this problem, if we had four distinct roots then we'd get 4 easy eq^n's from evaluating $(*)$ at those roots) We evaluate $(*)$ at $s=1$ to get another eq^n;

$$1 = 2A + 2B + C + D = 2A + 2 - 1 = 2A + 1 \therefore A = 0$$

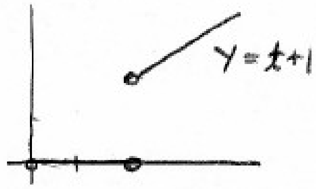
$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\}$$

$$= \boxed{t - \sin(t) = y(t)}$$

__Remark:__ Find the inverse transform is not usually easy!

§7.6 #6

$$g(t) = \begin{cases} 0 & 0<t<2 \\ t+1 & 2<t \end{cases} \quad \leftarrow \text{turn this on at } t>2.$$



$u(t-2)$ has what we want, it is zero for $t<2$ and it is one for $t>2$

thus $\boxed{g(t) = (t+1)\,u(t-2)}$

§7.6 #5 $g(t) = \begin{cases} 0 & 0<t<1 \\ 2 & 1<t<2 \\ 1 & 2<t<3 \\ 3 & 3<t \end{cases}$

$$g(t) = \underset{\substack{\uparrow \\ \text{turns on} \\ \text{the 2 at} \\ t=1}}{2u(t-1)} + \underset{\substack{\uparrow \\ \text{turns off the} \\ \text{2 and turns} \\ \text{on the 1} \\ \text{at } t=2}}{(1-2)u(t-2)} + \underset{\substack{\uparrow \\ \text{turns off the 1} \\ \text{turns on the 3} \\ \text{at } t=3}}{(3-1)u(t-3)}$$

$$\boxed{g(t) = 2u(t-1) - u(t-2) + 2u(t-3)}$$

Now compute the Laplace transform using Thm (8) which says $\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)$.

$$\begin{aligned}\mathcal{L}\{g(t)\}(s) &= 2\mathcal{L}\{u(t-1)\}(s) - \mathcal{L}\{u(t-2)\}(s) + 2\mathcal{L}\{u(t-3)\}(s) \\ &= \frac{2}{s}e^{-s} - \frac{1}{s}e^{-2s} + \frac{2}{s}e^{-3s} \qquad (\text{used } \mathcal{L}\{1\}(s) = \tfrac{1}{s}) \\ &= \boxed{\frac{1}{s}\left(2e^{-s} - e^{-2s} + 2e^{-3s}\right) = G(s)}\end{aligned}$$

§7.6 #6 Continued

$$\begin{aligned}\mathcal{L}\{g(t)\} &= \mathcal{L}\{(t+1)\,u(t-2)\}(s) \\ &= \mathcal{L}\{((t-2)+3)\,u(t-2)\}(s) \\ &= e^{-as}F(s) \quad \text{where } f(t) = t+3 \ \& \ a=2. \\ &= \boxed{e^{-2s}\left(\frac{1}{s^2} + \frac{3}{s}\right)}\end{aligned}$$

§7.6#9 $g(t) = \begin{cases} 0 & t<1 \\ t-1 & 1<t<2 \\ 3-t & 2<t<3 \\ 0 & 3<t \end{cases}$ } just fit lines to the graph

$$g(t) = (t-1)u(t-1) + [(3-t)-(t-1)]u(t-2) + [0-(3-t)]u(t-3)$$

$$\mathcal{L}\{g(t)\}(s) = \mathcal{L}\{(t-1)u(t-1)\} + \mathcal{L}\{(4-2t)u(t-2)\} + \mathcal{L}\{(t-3)u(t-3)\}$$
$$= \frac{1}{s^2}e^{-s} - \mathcal{L}\{2(t-2)u(t-2)\} + \frac{1}{s^2}e^{-3s}$$
$$= \frac{1}{s^2}e^{-s} - \frac{2}{s^2}e^{-2s} + \frac{1}{s^2}e^{-3s} = \boxed{\frac{1}{s^2}\left(e^{-s} - 2e^{-2s} + e^{-3s}\right)}$$

§7.6#12 Let $G(s) = e^{-3s}/s^2$ then recall Th^m(8) says that $\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a)$. Here $F(s) = 1/s^2 \Rightarrow f(t) = t$.

$$\mathcal{L}^{-1}\{e^{-3s}\tfrac{1}{s^2}\} = f(t-3)u(t-3) = \boxed{(t-3)u(t-3)}$$

§7.6#13

$$\mathcal{L}^{-1}\{e^{-2s}\frac{1}{s+2}\}(t) = e^{-2\tau}\Big|_{\tau=t-2} u(t-2) = e^{-2(t-2)}u(t-2)$$

$$\mathcal{L}^{-1}\{e^{-4s}\frac{1}{s+2}\}(t) = e^{-2(t-4)}u(t-4)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{e^{-2s}-3e^{-4s}}{s+2}\right\}(t) = \boxed{e^{-2(t-2)}u(t-2) - 3e^{-2(t-4)}u(t-4)}$$

§7.6#16 Note $\mathcal{L}^{-1}\{\frac{1}{s^2+4}\}(t) = \frac{1}{2}\sin(2t)$ thus

$$\mathcal{L}^{-1}\{e^{-s}\frac{1}{s^2+4}\}(t) = \boxed{\frac{1}{2}\sin(2(t-1))\,u(t-1)}$$

§7.6 #23

$$f(t) = \begin{cases} e^{-t} & 0<t<1 \\ 1 & 1<t<2 \end{cases}$$ with period $T=2$ so this repeats again & again & ...

The "windowed" version of $f(t)$ is just the function $f(t)$ for one period then zero elsewhere

$$f_T(t) = \begin{cases} f(t) & 0<t<2 \\ 0 & \text{other } t \end{cases}$$

Then Th^{m} (9) tells us how to transform such a function,

$$\mathcal{L}\{f\}(s) = \frac{F_T(s)}{1-e^{-sT}}$$

$$F_T(s) = \int_0^2 e^{-st} f(t) = \int_0^1 e^{-st}e^{-t}dt + \int_1^2 e^{-st}dt$$

$$= \int_0^1 e^{-(s+1)t}dt + \frac{-1}{s}e^{-st}\Big|_1^2$$

$$= \frac{-1}{s+1}e^{-(s+1)t}\Big|_0^1 - \frac{1}{s}(e^{-2s}-e^{-s})$$

$$= \frac{-1}{s+1}\left(e^{-(s+1)}-1\right) - \frac{1}{s}(e^{-2s}-e^{-s})$$

Therefore,

$$\boxed{\mathcal{L}\{f\}(s) = \frac{1}{1-e^{-2s}}\left\{\frac{1}{s+1}(1-e^{-s-1}) - \frac{1}{s}(e^{-2s}-e^{-s})\right\}}$$

- Alternatively we could use the unit-step th^{m}'s to compute $F_T(s)$

Notice $f_T(t) = e^{-t}u(t) + [1-e^{-t}]u(t-1) + [0-1]u(t-2)$ thus

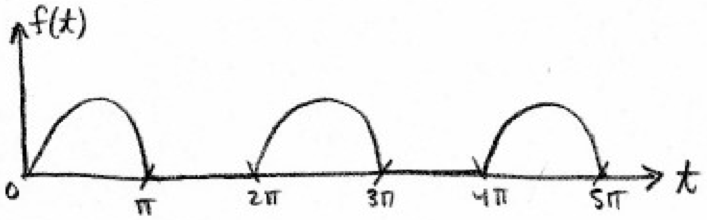
$$\mathcal{L}\{f_T\}(s) = \mathcal{L}\{e^{-t}u(t)\} + \mathcal{L}\{u(t-1)\} - \mathcal{L}\{e^{-1}e^{-(t-1)}u(t-1)\} - \mathcal{L}\{u(t-2)\}$$

$$= \frac{1}{s+1}e^{0\cdot s} + \frac{1}{s}e^{-s} - e^{-1}\frac{e^{-s}}{s+1} - \frac{1}{s}e^{-2s}$$

$$= \frac{1}{s+1}(1-e^{-s-1}) - \frac{1}{s}(e^{-2s}-e^{-s}) = F_T(s)$$

Guess it's about the same work either way.

§ 7.6 # 28



period is 2π.

$$f_T(t) = u(t)\sin(t) - u(t-\pi)\sin(t)$$

$$\mathcal{L}\{f_T\}(s) = \mathcal{L}\{\sin t \cdot u(t)\} - \mathcal{L}\{\sin(t-\pi+\pi)\ u(t-\pi)\}$$

$$= \frac{1}{s^2+1} e^{0s} - \mathcal{L}\{[\sin(t-\pi)\cos\pi + \cos(t-\pi)\underbrace{\sin\pi}_{0}]u(t-\pi)\}$$

$$= \frac{1}{s^2+1} + \mathcal{L}\{\sin(t-\pi)\,u(t-\pi)\}$$

$$= \frac{1}{s^2+1} + \frac{1}{s^2+1}e^{-\pi s} = F_T(s)$$

Thus we construct by Th^{m} (9),

$$\mathcal{L}\{f\}(s) = \frac{F_T(s)}{1-e^{-sT}} = \boxed{\frac{1}{1-e^{-2\pi s}}\left(\frac{1}{s^2+1} + \frac{1}{s^2+1}e^{-\pi s}\right)}$$

- Alternatively you could just calculate $F_T(s)$ straight from the def^{n} like I did in #23, I don't want to do those integrations though so instead I chose the route of writing f_T in terms of the unit-step function which allows me to use the Th^{m}(8) $\mathcal{L}\{f(t-a)\,u(t-a)\} = F(s)e^{-sa}$ to calculate.

§7.6 #34

$$y'' + 4y' + 4y = u(t-\pi) - u(t-2\pi) \equiv g(t) \quad \text{with } y(0)=0,\ y'(0)=0$$

Transform the eqⁿ to the "frequency domain",

$$s^2 Y + 4sY + 4Y = \frac{1}{s}e^{-\pi s} - \frac{1}{s}e^{-2\pi s}$$ (used intial conditions here)

$$Y(s) = \frac{1}{s^2+4s+4}\left(\frac{1}{s}\right)\left\{e^{-\pi s} - e^{-2\pi s}\right\}$$

$$= \frac{1}{(s+2)^2}\frac{1}{s}\left\{e^{-\pi s} - e^{-2\pi s}\right\}$$

partial fractions.

$$= \left(\frac{-1}{4(s+2)} - \frac{1}{2(s+2)^2} + \frac{1}{4s}\right)\left(e^{-\pi s} - e^{-2\pi s}\right)$$

Now to solve for y we must take the inverse Laplace transform. Use $\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)\,u(t-a)$

$$\mathcal{L}^{-1}\left\{\frac{-1}{4(s+2)} - \frac{1}{2(s+2)^2} + \frac{1}{4s}\right\}(t) = \frac{-1}{4}e^{-2t} - \frac{1}{2}te^{-2t} + \frac{1}{4} = f(t).$$

Thus

$$\mathcal{L}^{-1}\{Y\}(t) = f(t-\pi)\,u(t-\pi) - f(t-2\pi)\,u(t-2\pi)$$

$$= \boxed{\left(\frac{-1}{4}e^{-2(t-\pi)} - \frac{1}{2}(t-\pi)e^{-2(t-\pi)} + \frac{1}{4}\right)u(t-\pi) - \left(\frac{-1}{4}e^{-2(t-2\pi)} - \frac{1}{2}(t-2\pi)e^{-2(t-2\pi)} + \frac{1}{4}\right)u(t-2\pi)}$$

Comment: Solving such a problem w/o Laplace transforms is probably even more trouble. This type of eqⁿ is quite important to Electrical Engineering. However, not just that, in fact any application with piecewise defined inputs tend to be more naturally treated by the Laplace transform. The later sections in this chapter will show even more convincing proof that the Laplace transform is truly a powerful tool.

§7.7 #1 Reduce the problem to quadrature (a.k.a give the

Sol$^{\underline{n}}$ in terms of some integral). Use the convolution Th$^{\underline{m}}$ (11)

$$\mathcal{L}\{f*g\}(s) = F(s)\,G(s)$$ (assuming f & g are continous and of exponential order α)

$$\mathcal{L}^{-1}\{FG\}(t) = (f*g)(t)$$

Where we define the convolution of f & g by

$$(f*g)(t) \equiv \int_0^t f(t-v)\,g(v)\,dv$$

That's the theory. The problem is to solve $y'' - 2y' + y = g(t)$

Here $y(0) = -1$ and $y'(0) = 1$. Transform our eq$^{\underline{n}}$ to frequency domain,

$$s^2 Y + s - 1 - 2(sY - 1) + Y = \mathcal{L}\{g(t)\}(s) \equiv G(s)$$

$$Y(s^2 - 2s + 1) = 3 - s + G(s)$$

$$Y(s) = \frac{3-s}{s^2-2s+1} + \frac{1}{s^2-2s+1}G(s)$$

$$Y(s) = \frac{2}{(s-1)^2} - \frac{1}{s-1} + \frac{1}{(s-1)^2}G(s)$$

partial fractions

Then note $\mathcal{L}^{-1}\{1/(s-1)^2\}(t) = te^t$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}G(s)\right\} = (f*g)(t) = \int_0^t (t-v)e^{t-v}g(v)\,dv$$

Thus computing the first to terms inverse transform & adding 2

$$\mathcal{L}^{-1}\{Y\}(t) = \boxed{y(t) = 2te^t - e^t + \int_0^t (t-v)e^{t-v}g(v)\,dv}$$

§7.7 #4 $y'' + y = g(t) \qquad y(0) = 0 \ \$ \ y'(0) = 1$

$$s^2 \bar{Y} - s + \bar{Y} = G(s)$$

$$\bar{Y} = \frac{s}{s^2+1} + \frac{1}{s^2+1} G(s) \qquad F(s) = \frac{1}{s^2+1} \longrightarrow f(t) = \sin t$$

$$\boxed{y = \cos(t) + \int_0^t \sin(t-v)\, g(v)\, dv}$$

(Same logical flow as in #1)

§7.7 #5

$$\mathcal{L}^{-1}\Big\{\underbrace{\tfrac{1}{s}}_{F}\ \underbrace{\tfrac{1}{s^2+1}}_{G}\Big\}(t) = (f * g)(t) \quad \text{where} \quad \begin{aligned} f(t) &= \mathcal{L}^{-1}\{1/s\}(t) = 1 \\ g(t) &= \mathcal{L}^{-1}\Big\{\tfrac{1}{s^2+1}\Big\}(t) = \sin t \end{aligned}$$

$$(f*g)(t) = \int_0^t f(t-v)\, g(v)\, dv$$

$$= \int_0^t \sin(v)\, dv = -\cos(v)\Big|_0^t = \boxed{-\cos(t) + 1 = \mathcal{L}^{-1}\Big\{\tfrac{1}{s}\,\tfrac{1}{s^2+1}\Big\}(t)}$$

§7.7 #8

$$\mathcal{L}^{-1}\Big\{\frac{1}{s^2+4}\,\frac{1}{s^2+4}\Big\}(t) = \frac{1}{4}\sin(2t) * \sin(2t) \quad : \quad \mathcal{L}^{-1}\Big\{\frac{1}{s^2+4}\Big\} = \frac{1}{2}\sin 2t.$$

$$= \frac{1}{4}\int_0^t \sin(2t-2v)\sin 2v \, dv$$

$\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$

$$= \frac{1}{4}\int_0^t [\sin 2t \cos 2v - \sin 2v \cos 2t]\sin 2v \, dv$$

$$= \frac{1}{4}\sin 2t \int_0^t \sin 2v \cos 2v\, dv - \frac{\cos 2t}{4}\int_0^t \sin^2(2v)\, dv$$

$\sin(2\theta) = 2\sin\theta\cos\theta$
$\$ \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$= \frac{1}{8}\sin 2t \int_0^t \sin(4v)\,dv - \frac{1}{8}\cos(2t)\int_0^t [1-\cos(4v)]\,dv$$

$$= \frac{1}{8}\sin 2t \left(\frac{-1}{4}\cos 4v\Big|_0^t\right) - \frac{1}{8}\cos 2t\left(v - \frac{1}{4}\sin 4v\Big|_0^t\right)$$

$$= \boxed{\frac{1}{32}\sin(2t)(1-\cos 4t) - \frac{1}{8}t\cos 2t + \frac{1}{32}\cos 2t \sin 4t}$$

§7.7 #11

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-1)(s+2)}\right\}(t) = \mathcal{L}^{-1}\left\{\left(1+\frac{1}{s-1}\right)\left(\frac{1}{s+2}\right)\right\}(t)$$

$$= \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t)}_{e^{-2t}} + \mathcal{L}^{-1}\left\{\underbrace{\left(\frac{1}{s-1}\right)}_{F}\underbrace{\left(\frac{1}{s+2}\right)}_{G}\right\}(t) \Rightarrow \begin{array}{l} f(t)=e^{t} \\ g(t)=e^{-2t}\end{array}$$

$$\mathcal{L}^{-1}\{FG\}(t) = (f*g)(t) = \int_0^t f(t-v)g(v)\,dv$$

$$= \int_0^t e^{t-v}e^{-2v}\,dv$$

$$= \int_0^t e^{t-3v}\,dv$$

$$= \frac{-1}{3}e^{t-3v}\Big|_0^t = \left(e^{-2t} - e^{t}\right)\left(\frac{-1}{3}\right)$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{s}{(s-1)(s+2)}\right\}(t) = e^{-2t} - \frac{1}{3}e^{-2t} + \frac{1}{3}e^{t} = \boxed{\frac{2}{3}e^{-2t} + \frac{1}{3}e^{t}}$$

§7.7 #14

$$f(t) = \int_0^t e^{v}\sin(t-v)\,dv = (\overset{g(t)}{\sin(t)} * \overset{h(t)}{e^{t}})(t) = (g*h)(t)$$

$$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{g*h\}(s) = G(s)\,H(s)$$

$$G(s) = \mathcal{L}\{\sin t\}(s) = \frac{1}{s^2+1}$$

$$H(s) = \mathcal{L}\{e^{t}\}(s) = \frac{1}{s-1} \qquad \therefore \boxed{\mathcal{L}\{f\}(s) = \left(\frac{1}{s^2+1}\right)\left(\frac{1}{s-1}\right)}$$

§7.8 #1

$$\int_{-\infty}^{\infty} (t^2-1)\,\delta(t)\,dt = (t^2-1)\Big|_{t=0} = \boxed{-1}$$

§7.8 #2

$$\int_{-\infty}^{\infty} e^{3t}\,\delta(t)\,dt = e^{3t}\Big|_{t=0} = e^{3(0)} = \boxed{1}$$

§7.8 #3

$$\int_{-\infty}^{\infty} \sin(3t)\,\delta(t-\pi/2)\,dt = \sin(3t)\Big|_{t=\pi/2} = \sin\left(\tfrac{3\pi}{2}\right) = \boxed{-1}$$

§7.8 #4

$$\int_{-\infty}^{\infty} e^{-2t}\,\delta(t+1)\,dt = e^{-2t}\Big|_{t=-1} = e^{-2(-1)} = \boxed{e^2}$$

§7.8 #7

$\mathcal{L}\{\delta(t-1)\}(s) = e^{-s}$ using eq^{n}(6) of p. 409.

$\mathcal{L}\{\delta(t-3)\}(s) = e^{-3s}$

Thus $\mathcal{L}\{\delta(t-1)-\delta(t-3)\}(s) = \boxed{e^{-s}-e^{-3s}}$

§7.8 #11

$$\mathcal{L}\{\delta(t-\pi)\sin t\} = \int_0^{\infty} \delta(t-\pi)\sin t\, e^{-st}\,dt \leftarrow \text{def}^{n} \text{ of Laplace Transform}$$

$$= (\sin t)e^{-st}\Big|_{t=\pi}$$

$$= (\sin \pi)e^{-s\pi}$$

$$= \boxed{0}$$

<u>Remark</u>: δ-function makes integrals become evaluations instead, much easier!

§ 7.8 #13 $w'' + w = \delta(t-\pi)$ $w(0)=0$ & $w'(0)=0$

$$s^2 W(s) + W(s) = \mathcal{L}\{\delta(t-\pi)\}(s) = e^{-\pi s}$$ ← (used intial conditions)

$$W(s)(s^2+1) = e^{-\pi s} \Rightarrow W(s) = \frac{1}{s^2+1} e^{-\pi s}$$

Now to solve for $w(t)$ we take inverse Laplace transform,

$$w(t) = \mathcal{L}^{-1}\{W\}(t) = \mathcal{L}^{-1}\{e^{-\pi s} \underbrace{\frac{1}{s^2+1}}_{F(s)}\} \quad : \quad f(t) = \sin(t)$$

$$= f(t-a)\,u(t-a)$$: see Th^{m} (8) p. 387.

$$= \sin(t-\pi)\,u(t-\pi)$$

$$= (\sin t \cos\pi - \sin\pi \cos t)\,u(t-\pi)$$

$$= \boxed{-\sin(t)\,u(t-\pi)}$$

Similar to #17

$$y'' + y = 4\delta(t-2) + t^2 \qquad y(0)=0, \quad y'(0)=2$$

$$s^2 Y - 2 + Y = 4e^{-2s} + \frac{2}{s^3}$$

$$Y(s^2+1) = 2 + 4e^{-2s} + 2/s^3$$

$$Y = \frac{2}{s^2+1} + \frac{4}{s^2+1}e^{-2s} + \underbrace{\frac{2}{(s^2+1)s^3}}_{\boxed{\frac{s}{s^2+1} - \frac{1}{s} + \frac{1}{s^3}}}$$

partial fractions

$$\boxed{y = 2\sin(t) + 4\sin(t-2)\,u(t-2) + \cos t - 1 + \tfrac{1}{2}t^2}$$

§7.8 #29 The following eq^{n}'s model a mass on a spring that is struck by a hammer at $t = \pi/2$.

$$\frac{d^2x}{dt^2} + 9x = -3\delta(t-\pi/2) \quad \text{with } x(0) = 1 \ \& \ x'(0) = 0$$

Taking the Laplace transform yields,

$$s^2 X - s + 9X = -3e^{-\frac{\pi s}{2}}$$

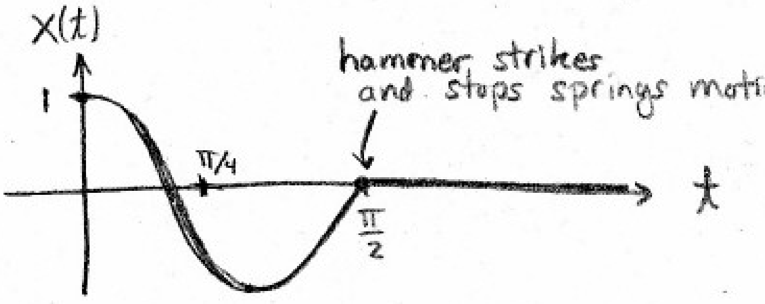
$$X(s^2+9) = s - 3e^{-\pi s/2}$$

$$X(s) = \frac{s}{s^2+9} - \frac{3}{s^2+9}e^{-\pi s/2}$$

$$\begin{aligned} x(t) = \mathcal{L}^{-1}\{X\}(t) &= \cos(3t) - \sin(3(t-\pi/2))\,u(t-\pi/2) \\ &= \cos(3t) - \sin(3t - 3\pi/2)\,u(t-\pi/2) \\ &= \cos(3t) - [\sin(3t)\cos(3\pi/2) - \sin(\tfrac{3\pi}{2})\cos(3t)]\,u(t-\pi/2) \\ &= \cos(3t) - \cos(3t)\,u(t-\pi/2) \\ &= \boxed{\cos(3t)\,[1 - u(t-\pi/2)] = x(t)} \end{aligned}$$

$$x(t) = \begin{cases} \cos(3t) & \text{for } t < \pi/2 \\ 0 & \text{for } t > \pi/2 \end{cases}$$

the mass ceases moving after being hit by the hammer.



Remark: we could also understand earlier problems to describe a similar physical situation, for example #13 had $w(t) = -\sin(t)\,u(t-\pi)$ with $w(0) = w'(0) = 0$ this means the hammer struck the spring and set it vibrating for $t > \pi$. The possibilities are endless, but the fact the δ-function models such impulses is neat.

§7.9 #1 $x(0) = 1$ and $y(0) = 1$

$$x' = 3x - 2y \Rightarrow s\mathcal{X} - 1 = 3\mathcal{X} - 2\mathcal{Y}$$
$$y' = 3y - 2x \Rightarrow s\mathcal{Y} - 1 = 3\mathcal{Y} - 2\mathcal{X}$$

$$\left.\begin{aligned}(s-3)\mathcal{X} + 2\mathcal{Y} &= 1 \\ (s-3)\mathcal{Y} + 2\mathcal{X} &= 1\end{aligned}\right\} \Rightarrow \begin{bmatrix} s-3 & 2 \\ 2 & s-3 \end{bmatrix}\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Use Kramer's Rule, (see A5-A6 in the appendix)

$$\mathcal{X} = \frac{\det\begin{vmatrix} 1 & 2 \\ 1 & s-3 \end{vmatrix}}{\det\begin{vmatrix} s-3 & 2 \\ 2 & s-3 \end{vmatrix}} = \frac{s-3-2}{(s-3)^2-4} = \frac{s-5}{s^2-6s+5} = \frac{1}{s-1}$$

$$\mathcal{Y} = \frac{\det\begin{vmatrix} s-3 & 1 \\ 2 & 1 \end{vmatrix}}{\det\begin{vmatrix} s-3 & 2 \\ 2 & s-3 \end{vmatrix}} = \frac{s-3-2}{(s-3)^2-4} = \frac{1}{s-1}$$

Then as $\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) = e^t$ thus $\boxed{x(t) = e^t \text{ \& } y(t) = e^t}$

Remark: you could also solve by substitution, I just prefer matrix methods.