§9.1 #11

(1) $x'' + 3x + 2y = 0$

(2) $y'' - 2x = 0$

Let $x_1 = x$, $x_2 = x'$, $x_3 = y$ and $x_4 = y'$

(1) $\Rightarrow x_2' + 3x_1 + 2x_3 = 0$

(2) $\Rightarrow x_4' - 2x_1 = 0$

Also $x_2 = x' = x_1'$ and $x_4 = y' = x_3'$. Lets rewrite everything

$$\begin{cases} x_1' = x_2 \\ x_2' = -3x_1 - 2x_3 \\ x_3' = x_4 \\ x_4' = 2x_1 \end{cases} \longleftrightarrow \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}$$

matrix normal form.

§9.2 #5

$-x_1 + 2x_2 = 0 \Rightarrow x_1 = 2x_2$

$2x_1 + 3x_2 = 0 \Rightarrow 2(2x_2) + 3x_2 = 7x_2 = 0 \Rightarrow \boxed{x_2 = 0}$

$\Rightarrow \boxed{x_1 = 0}$

§9.2 #8

$x_1 + x_2 - x_3 = 0 \Rightarrow x_1 = x_3 - x_2$ eliminated x_1

$-x_1 - x_2 + x_3 = 0 \Rightarrow x_2 - x_3 - x_2 + x_3 = 0$ (no info)

$x_1 + x_2 - x_3 = 0 \Rightarrow x_3 - x_2 + x_2 - x_3 = 0$ (no info)

These 3 eqⁿ's are the same, $x_1 = x_3 - x_2$ has infinitely many solⁿ's. For example

$x_1 = x_2 = x_3 = 0$ is a solⁿ.

$x_1 = 4$ and $x_3 = 6$, $x_2 = 2$

$x_1 = 5$ and $x_3 = 7$, $x_2 = 2$ and so on... to be precise let $x_1 = s$ and $x_2 = t$ then $x_3 = x_1 + x_2 = s + t$

§9.2 #7

$-x_1 + 3x_2 = 0$ Eqⁿ(1)

$-3x_1 + 9x_2 = 0$ Eqⁿ(2)

Eqⁿ(1) $\Rightarrow x_1 = 3x_2$ then substitute into Eqⁿ(2) $-3(3x_2) + 9x_2 = 0$ no data gained, just $0 = 0$. Really Eqⁿ(1) & Eqⁿ(2) are the same. There are ∞ly many solⁿ's.

$$\boxed{x_1 = s \text{ then } x_2 = \tfrac{1}{3}s}$$

§9.3#3 $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -2+20 & 6+8 \\ -1+5 & 3+2 \end{bmatrix} = \boxed{\begin{bmatrix} 18 & 14 \\ 4 & 5 \end{bmatrix} = AB}$$

$$A^2 = AA = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4+4 & 8+4 \\ 2+1 & 4+1 \end{bmatrix} = \boxed{\begin{bmatrix} 8 & 12 \\ 3 & 5 \end{bmatrix} = A^2}$$

§9.3#4 $A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 1 \\ 0 & 4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2+3 & -2+1 \\ 0 & 0+12 & 4 \\ -1 & -1+9 & 1+3 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 5 & -1 \\ 0 & 12 & 4 \\ -1 & 8 & 4 \end{bmatrix} = AB}$$

§9.3#7 The transpose of a matrix A is obtained by switching rows for columns ; in components if a $(m \times n)$ matrix A has components A_{ij} then $(A^T)_{ij} = A_{ji}$ meaning A^T will be an $(n \times m)$ matrix.

a.) $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ both $(n \times 1)$ matrices we call column vectors.

$$u^T v = (u_1 \; u_2 \cdots u_n)\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = u \cdot v$$

viewing the column vectors as vectors in the geometric sense.

d.) Yes because $(AB)_{ij} = \sum_k A_{ik} B_{kj}$

So we can prove it,

$$(AB)^T_{ij} = \sum_k A_{jk} B_{ki}$$
$$= \sum_k B_{ki} A_{jk}$$
#'s commute.
$$= \sum_k (B^T)_{ik} (A^T)_{kj} \equiv (B^T A^T)_{ij}$$

Ok, I'm misbehaving a little here, this is more for ma 405 not here ☹

§9.3 #9 The inverse of a 2×2 matrix (when there is one) is easy to calculate because for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we can write

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ (general formula for 2×2 case)

It's simple to check that this works,

$$AA^{-1} = \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This only makes sense when $ad-bc \neq 0$. The quantity $ad-bc = \det(A)$, it is very important!

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}^{-1} = \frac{1}{8+1}\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 4/9 & -1/9 \\ 1/9 & 2/9 \end{bmatrix}}$$

§9.3 #12 No such easy formula exists for 3×3 case. We'll use the algorithm $[A|I] \longrightarrow [I|A^{-1}]$ where the $\longrightarrow$ involves row operations.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} \text{(unchanged)} \\ \text{(subtracted R1)} \\ \text{(sub. 2R1)} \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right] \begin{array}{l} \text{(sub. R2)} \\ \text{(unchanged)} \\ \text{(sub R2)} \end{array}$$

Hey, wait a minute this isn't going to work. notice that $\det(A) = 0$ ∴ A^{-1} doesn't exist!

§9.3 #14 this time I checked $\det\begin{vmatrix} 1 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 3 \Rightarrow A^{-1}$ exists, lets find it,

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 3 & 3 & 3 & 3 & 0 & 0 \\ 0 & 3 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 3 & 0 & 1 & 2 & -1 & 0 \\ 0 & 3 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 3 & -1 & -1 \\ 0 & 3 & 0 & 3 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/3 & -1/3 \\ 0 & 1 & 0 & 1 & 1/3 & -2/3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Thus $$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} 1 & -1/3 & -1/3 \\ 1 & 1/3 & -2/3 \\ -1 & 0 & 1 \end{bmatrix}}$$

Remark: Use technology to do this! grr...

§9.3 # 18

$$\Psi(t) = \begin{bmatrix} \sin 2t & \cos 2t \\ 2\cos 2t & -2\sin 2t \end{bmatrix}$$

$$\Psi^{-1}(t) = \frac{1}{-2\sin^2 2t - 2\cos^2 2t}\begin{bmatrix} -2\sin 2t & -\cos 2t \\ -2\cos 2t & \sin 2t \end{bmatrix}$$ (general 2x2 inverse f-la)

$$\boxed{\Psi^{-1}(t) = \begin{bmatrix} \sin 2t & \frac{1}{2}\cos 2t \\ \cos 2t & -\frac{1}{2}\sin 2t \end{bmatrix}}$$

§9.3 # 20 Here I use a popular algorithm to find the inverse, note the row operations listed below ↓

$$\underbrace{\begin{bmatrix} e^{3t} & 1 & t \\ 3e^{3t} & 0 & 1 \\ 9e^{3t} & 0 & 0 \end{bmatrix}}_{\Psi(t)}\left|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right] \sim \begin{bmatrix} e^{3t} & 1 & t \\ 0 & -3 & 1-3t \\ 0 & -9 & -9t \end{bmatrix}\left|\begin{matrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -9 & 0 & 1 \end{matrix}\right] \quad \begin{matrix} R1 \\ R2-3R1 \\ R3-9R1 \end{matrix}$$

$$\sim \begin{bmatrix} 3e^{3t} & 3 & 3t \\ 0 & -3 & 1-3t \\ 0 & -9 & -9t \end{bmatrix}\left|\begin{matrix} 3 & 0 & 0 \\ -3 & 1 & 0 \\ -9 & 0 & 1 \end{matrix}\right] \quad \begin{matrix} 3R1 \\ R2 \\ R3 \end{matrix}$$

$$\sim \begin{bmatrix} 3e^{3t} & 0 & 1 \\ 0 & -3 & 1-3t \\ 0 & 0 & -3 \end{bmatrix}\left|\begin{matrix} 0 & 1 & 0 \\ -3 & 1 & 0 \\ 0 & -3 & 1 \end{matrix}\right] \quad \begin{matrix} R1+R2 \\ R2 \\ R3-3R2 \end{matrix}$$

$$\sim \begin{bmatrix} 3e^{3t} & 0 & 1 \\ 0 & -3 & 1-3t \\ 0 & 0 & 1 \end{bmatrix}\left|\begin{matrix} 0 & 1 & 0 \\ -3 & 1 & 0 \\ 0 & 1 & -1/3 \end{matrix}\right] \quad \begin{matrix} R1 \\ R2 \\ R3/-3 \end{matrix}$$

$$\sim \begin{bmatrix} 3e^{3t} & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left|\begin{matrix} 0 & 0 & 1/3 \\ -3 & 3t & \frac{1}{3}(1-3t) \\ 0 & 1 & -1/3 \end{matrix}\right] \quad \begin{matrix} R1-R3 \\ R2-(1-3t)R3 \\ R3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\underbrace{\left|\begin{matrix} 0 & 0 & e^{-3t}/9 \\ 1 & -t & -(1-3t)/9 \\ 0 & 1 & -1/3 \end{matrix}\right]}_{\Psi^{-1}(t)} \quad \begin{matrix} R1/3e^{3t} \\ R2/-3 \\ R3 \end{matrix}$$

§9.3 # 24

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 3 & -1 \\ -1 & 2 & 1 \end{vmatrix} = (1)\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} - (0)\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} + 2\begin{vmatrix} 0 & 3 \\ -1 & 2 \end{vmatrix}$$

$$= (3+2) + 2(3)$$

$$= \boxed{11}$$

§9.3 # 37

$$\Phi(t) = \begin{bmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{bmatrix} \Rightarrow \Phi'(t) = \begin{bmatrix} 2e^{2t} & 3e^{3t} \\ -2e^{2t} & -6e^{3t} \end{bmatrix}$$

differentiate component-wise.

$$\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\Phi = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} & 3e^{3t} \\ -2e^{2t} & -6e^{3t} \end{bmatrix}$$

Thus $\Phi'(t) = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\Phi(t)$ thus the given $\Phi(t)$ is a sol^n to the matrix differential eq^n.

§9.3 # 40 $A(t) = \begin{bmatrix} 1 & e^{-2t} \\ 3 & e^{-2t} \end{bmatrix}$ & $B(t) = \begin{bmatrix} e^{-t} & e^{-t} \\ -e^{-t} & 3e^{-t} \end{bmatrix}$

a.) $$\int A(t)dt = \begin{bmatrix} \int dt & \int e^{-2t}dt \\ \int 3dt & \int e^{-2t}dt \end{bmatrix} = \begin{bmatrix} t + c_1 & -\frac{1}{2}e^{-2t} + c_2 \\ 3t + c_3 & -\frac{1}{2}e^{-2t} + c_4 \end{bmatrix}$$

$$\Rightarrow \boxed{\int A(t)dt = \begin{bmatrix} t & -\frac{1}{2}\exp(-2t) \\ 3t & -\frac{1}{2}\exp(-2t) \end{bmatrix} + C} \qquad C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$$

b.) $$\int_0^1 B(t)\,dt = \begin{bmatrix} \int_0^1 e^{-t}dt & \int_0^1 e^{-t}dt \\ \int_0^1 -e^{-t}dt & \int_0^1 3e^{-t}dt \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-1}+1 & -e^{-1}+1 \\ e^{-1}-1 & 3(-e^{-1}+1) \end{bmatrix}$$

$$= (-e^{-1}+1)\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

Easier way would be $B(t) = e^{-t}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ at begin of calculation then

$$\int_0^1 B(t)dt = \int_0^1 e^{-t}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}dt = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\int_0^1 e^{-t}dt = (-e^{-1}+1)\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}.$$

c.) $$\frac{d}{dt}\big(A(t)B(t)\big) = \frac{dA}{dt}B + A\frac{dB}{dt}$$ note $\frac{dB}{dt} = \frac{d}{dt}\left(e^{-t}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\right) = -B$

$$= \begin{bmatrix} 0 & -2e^{-2t} \\ 0 & -2e^{-2t} \end{bmatrix}B + \begin{bmatrix} 1 & e^{-2t} \\ 3 & e^{-2t} \end{bmatrix}(-B)$$

$$= \left(\begin{bmatrix} 0 & -2e^{-2t} \\ 0 & -2e^{-2t} \end{bmatrix} - \begin{bmatrix} 1 & e^{-2t} \\ 3 & e^{-2t} \end{bmatrix}\right)e^{-t}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

should simplify this, but you can do it.

(§9.4 #3)

$$\frac{dx}{dt} = t^2x - y - z + t$$
$$\frac{dy}{dt} = e^t z + 5$$
$$\frac{dz}{dt} = tx - y + 3z - e^t$$

$$\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}'}_{(\vec{x})'} = \underbrace{\begin{bmatrix} t^2 & -1 & -1 \\ 0 & 0 & e^t \\ t & -1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} t \\ 5 \\ -e^t \end{bmatrix}}_{\vec{f}}$$

(§9.4 #5) $y'' - 3y' - 10y = \sin(t)$

Let $x_1 = y$ and $x_2 = y'$ then

$$x_1' = x_2$$
$$x_2' = 3x_2 + 10x_1 + \sin t$$

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'}_{\vec{x}'} = \underbrace{\begin{bmatrix} 0 & 1 \\ 10 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ \sin t \end{bmatrix}}_{\vec{f}}$$

(§9.4 #7) $w''''(t) + w = t^2$

$w' = x$

$w'' = y \Rightarrow x' = y$

$w''' = z \Rightarrow z' + w = t^2 \Rightarrow z' = -w - t^2$ also $y' = z$

Why should I have to use $x_1, x_2, x_3, \ldots$ can use other letters.

$$\underbrace{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}'}_{(\vec{x})'} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ -t^2 \end{bmatrix}}_{\vec{f}} \approx \frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$$

(§9.4 #18)

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}, \begin{bmatrix} t^2 \\ 0 \\ t^2 \end{bmatrix}$ suppose $c_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 t^2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0$

$$\Rightarrow (c_1 + c_2 t + c_3 t^2)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$$
$$\Rightarrow c_1 + c_2 t + c_3 t^2 = 0$$
$$\Rightarrow \underline{c_1 = c_2 = c_3 = 0} \quad \text{(Equating Coefficients)}$$

∴ they are linearly independent.

§9.4#19 Suppose that X_1 & X_2 are sol^{n}'s to $x'(t) = Ax(t)$, where

$$X_1 = e^{2t}\begin{bmatrix}1\\-2\end{bmatrix} \quad \text{and} \quad X_2 = e^{2t}\begin{bmatrix}-2\\4\end{bmatrix}$$

$$W[x_1, x_2] = \begin{vmatrix} e^{2t} & -2e^{2t} \\ -2e^{2t} & 4e^{2t} \end{vmatrix} = 4e^{4t} - 4e^{4t} = 0$$

These are <u>not</u> linearly independent, in fact $X_2 = -2X_1$.
Thus these don't form a fundamental sol^{n} set.

§9.4#20 As in #19 except $X_1 = e^{-t}\begin{bmatrix}3\\2\end{bmatrix}$ and $X_2 = e^{4t}\begin{bmatrix}1\\-1\end{bmatrix}$

$$W[X_1, X_2] = \begin{vmatrix} 3e^{-t} & e^{4t} \\ 2e^{-t} & -e^{4t} \end{vmatrix} = -3e^{3t} - 2e^{3t} = -5e^{3t} \neq 0 \quad \forall t.$$

Thus $\{X_1, X_2\}$ is a fundamental sol^{n} set to the system $x'(t) = Ax(t)$. They give the general sol^{n}

$$x(t) = c_1 X_1 + c_2 X_2 = \boxed{c_1 e^{-t}\begin{bmatrix}3\\2\end{bmatrix} + c_2 e^{4t}\begin{bmatrix}1\\-1\end{bmatrix} = x(t)}$$

§9.4#22

$$W[X_1, X_2, X_3] = \begin{vmatrix} e^t & \sin t & -\cos t \\ e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \end{vmatrix}$$

$$= e^t \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} - \sin t \begin{vmatrix} e^t & \sin t \\ e^t & \cos t \end{vmatrix} - \cos t \begin{vmatrix} e^t & \cos t \\ e^t & -\sin t \end{vmatrix}$$

$$= e^t(\cos^2 t + \sin^2 t) - \sin t(\cos t - \sin t)e^t - \cos t(-\sin t - \cos t)e^t$$

$$= e^t(2 - \sin t \cos t + \cos t \sin t)$$

$= 2e^t \neq 0 \quad \forall t \in (-\infty, \infty)$. Thus $\{X_1, X_2, X_3\}$ forms fundamental sol^{n} set, they're L.I. and are assumed to solve $X' = AX$.

$$\boxed{X = c_1 e^t \begin{pmatrix}1\\1\\1\end{pmatrix} + c_2 \begin{pmatrix}\sin t\\ \cos t\\ -\sin t\end{pmatrix} + c_3 \begin{pmatrix}-\cos t\\ \sin t\\ \cos t\end{pmatrix}}$$

§9.4 #26 Let $\Phi(t) = [x_1(t) | x_2(t) | \cdots | x_n(t)]$ fundamental matrix is made from concatenating the n-linearly independent fundamental sol^{n}'s to the eq^{n} $x' = Ax$. Show that $x(t) = \Phi(t)\Phi^{-1}(t_o)x_o$ is the sol^{n} to the initial value problem $x' = Ax$ with $x(t_o) = x_o$

$$\frac{d}{dt}[x(t)] = \frac{d}{dt}\left[\Phi(t)\Phi^{-1}(t_o)x_o\right]$$
$$= \frac{d\Phi}{dt}\Phi^{-1}(t_o)x_o$$
$$= A\Phi(t)\Phi^{-1}(t_o)x_o$$
$$= Ax(t) \qquad \therefore\ x(t) = \Phi(t)\Phi^{-1}(t_o)x_o \text{ is a sol}^{n} \text{ of } x' = Ax.$$

Then $x(t_o) = \Phi(t_o)\Phi^{-1}(t_o)x_o = Ix_o = x_o$ as claimed.

§9.4 #28 $x' = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} x$ with $x(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$$\Phi(t) = \begin{bmatrix} e^{-t} & e^{5t} \\ -e^{-t} & e^{5t} \end{bmatrix} \longrightarrow \Phi^{-1}(t) = \frac{1}{2e^{4t}}\begin{bmatrix} e^{5t} & -e^{5t} \\ e^{-t} & e^{-t} \end{bmatrix}$$

Hence $x(t) = \Phi(t)\Phi^{-1}(t_o)x_o \quad (t_o = 0 \text{ here})$

$$= \Phi(t)\,\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} e^{-t} & e^{5t} \\ -e^{-t} & e^{5t} \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$
$$= \boxed{\begin{bmatrix} 2e^{-t} + e^{5t} \\ -2e^{-t} + e^{5t} \end{bmatrix} = x(t)}$$

Remark: in §9.4 and before the sol^{n}'s $x_1, x_2, \ldots, x_n$ to $x' = Ax$ just fall from the sky. In the upcoming sections. we will learn how to derive them using matrix techniques.

§9.5 #1 Find eigenvalues and eigenvectors of $\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} = A$ that is find λ and u such that $Au = \lambda u$ meaning,

$$0 = |A - \lambda I| = \begin{vmatrix} -4-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = (\lambda+4)(\lambda+1) - 4 = \lambda^2 + 5\lambda = \lambda(\lambda+5) = 0$$

Thus the eigenvalues of A are $\boxed{\lambda_1 = 0}$ and $\boxed{\lambda_2 = -5}$

- Now find u with $Au = 0$,

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -4u_1 + 2u_2 \\ 2u_1 - u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u_2 = 2u_1$$

$$\Rightarrow u = \begin{pmatrix} u_1 \\ 2u_1 \end{pmatrix} = u_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

It is customary to normalize the eigenvector to length 1 this gives $\hat{u} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Clearly any non-zero multiple of $\hat{u}$ will also be an eigenvector of $A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ with eigenvalue zero.

- Next find v with $Av = -5v$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -5\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{aligned} -4v_1 + 2v_2 &= -5v_1 \\ 2v_1 - v_2 &= -5v_2 \end{aligned} \Rightarrow \begin{aligned} -4v_1 + 2v_2 &= 5v_1 \\ 4v_1 - 2v_2 &= -10v_2 \end{aligned}$$

$$5v_1 - 10v_2 = 0 \quad \therefore \; v_1 = 2v_2$$

$$\therefore \boxed{v = v_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

$$\therefore \boxed{\hat{v} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

Let me write the answers more clearly.

$\lambda_1 = 0$ with $u = \begin{pmatrix} s \\ 2s \end{pmatrix}$ for $s \neq 0$.

$\lambda_2 = -5$ with $v = \begin{pmatrix} 2t \\ t \end{pmatrix}$ for $t \neq 0$ ← can't allow $s = 0$ or $t = 0$ because eigenvectors are by def$^{\underline{n}}$ not the zero vector.

↑ eigenvalues

↑ eigenvectors (there are ∞ly many of these for each eigenvalue.)

§9.5#4

$A = \begin{bmatrix} 1 & 5 \\ 1 & -3 \end{bmatrix}$ study $\begin{vmatrix} 1-\lambda & 5 \\ 1 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda) - 5$

$= -3 + 3\lambda - \lambda + \lambda^2 - 5$

$= \lambda^2 + 2\lambda - 8$

$= (\lambda+4)(\lambda-2) = 0 \quad \therefore \lambda_1 = -4, \ \lambda_2 = 2$

$\lambda_1 = -4$: $\begin{bmatrix} 1 & 5 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -4\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$u_1 + 5u_2 = -4u_1 \Rightarrow 5u_1 = -5u_2$

$\Rightarrow u_1 = -u_2$

$\Rightarrow \boxed{u = \begin{pmatrix} -s \\ s \end{pmatrix} \text{ with } \lambda_1 = -4}$

$\lambda_2 = 2$: $\begin{bmatrix} 1 & 5 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$v_1 + 5v_2 = 2v_1 \Rightarrow v_1 = 5v_2 \Rightarrow \boxed{v = \begin{pmatrix} 5s \\ s \end{pmatrix} \text{ with } \lambda_2 = 2}$

§9.5#6

$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda)$ since $\lambda^2 - 1 = (\lambda+1)(\lambda-1)$

$= [\lambda+1](-\lambda(\lambda-1) + 1 + 1)$

$= (\lambda+1)(-\lambda^2 + \lambda + 2)$

$= -(\lambda+1)(\lambda^2 - \lambda - 2)$

$= -(\lambda+1)(\lambda-2)(\lambda+1) \quad \therefore \lambda_1 = -1, \ \lambda_2 = 2, \ \lambda_3 = -1$ (λ_1, λ_3 repeated.)

§9.5#6 Continued, let $u^T = (x, y, z)$ for $Au = 2u$,

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} \rightarrow \begin{matrix} y+z = 2x \\ x+z = 2y \\ x+y = 2z \end{matrix}$$

$\rightarrow z = 2x - y = 2y - x \Rightarrow 3x = 3y \Rightarrow x = y$

$x = y$: $\rightarrow 2x = 2z \Rightarrow x = z$

Thus $u = \begin{pmatrix} s \\ s \\ s \end{pmatrix}$ for $\lambda_2 = 2$

The repeated root of $\lambda_1 = \lambda_3 = -1$ ought to give just one class of eigenvector. $Au = -u$ again use $u^T = (x, y, z)$ for convenience,

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y+z \\ x+z \\ x+y \end{bmatrix} = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix} \rightarrow x + y = -z$$

With x, y, z we have 3 degrees of freedom, but here we really have only one independent eq^n. This eq^n removes one degree of freedom but two will remain. Let $x = s$ and $y = t$ where s & t are arbitrairy parameters,

$$z = -x - y = -s - t$$

Then,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ t \\ -s-t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Just to be safe we'll check that these work,

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1-1 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ -1 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Thus we have explicitly shown that $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ are eigenvectors with eigenvalue $\lambda = -1$ with respect to the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

§9.5 #7

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 1 \\ 0 & 2 & 4-\lambda \end{vmatrix} = (1-\lambda)\left[(3-\lambda)(4-\lambda)-2\right]$$

$$= (1-\lambda)(12 - 7\lambda + \lambda^2 - 2)$$

$$= (1-\lambda)(\lambda^2 - 7\lambda + 10)$$

$$= (1-\lambda)(\lambda-2)(\lambda-5) = 0 \Rightarrow \underline{\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 5}$$

- To begin find eigenvector with $\lambda_1 = 1$,

$$Au = u \leftrightarrow (A-I)u = 0 \leftrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (*)$$

Use row operations
$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 0 & 2 & 3 \end{bmatrix} \Rightarrow \begin{matrix} 2u_1 - 2u_3 = 0 \\ 2u_2 + 3u_3 = 0 \end{matrix}$$

Let $u_1 = t$ then $u_3 = u_1 = t$ and $u_2 = \frac{-3}{2}u_3 = -\frac{3}{2}t$. Thus,

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} t \\ -3/2\, t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -3/2 \\ 1 \end{bmatrix}$$ eigenvector for each $t \neq 0$ with eigenvalue $\lambda_1 = 1$

- Next find u with $Au = 2u$ begin with (*)'s analogy for $\lambda_2 = 2$

$$\begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{matrix} u_1 = 0 \\ u_2 + u_3 = 0 \end{matrix}$$ let $u_2 = t$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ -t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

for $t \neq 0$ gives e.vect. with e.v. 2.

- Next find u with $Au = 5u$

$$\begin{bmatrix} -4 & 0 & 0 \\ 2 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \left\{ \begin{matrix} u_1 = 0 \\ -2u_2 + u_3 = 0 \end{matrix} \right\}$$

Let $u_2 = t$ then $u_3 = 2t$ thus

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 2t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$ eigenvector of $\lambda_3 = 5$ w.r.t. $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{bmatrix}$ (assuming $t \neq 0$)

Remark: Ive used matrix techniques to calculate the eigenvectors this is efficient but not necessary. You don't absolutely need to learn it, but it helps make things quicker.

§9.5 #9 find eigenvalues & vectors for $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$Au = \lambda u \iff (A - I\lambda)u = 0 \iff \det(A - I\lambda) = 0$$

This is why we always begin with the determinant equation; it gives us what λ must be for $Au = \lambda u$.

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \quad \therefore \quad \boxed{\lambda = \pm i}$$

Find $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ with $Au = iu$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} = i\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} iu_1 \\ iu_2 \end{bmatrix} \longrightarrow \begin{matrix} -u_2 = iu_1 \\ u_1 = iu_2 \end{matrix}$$

Thus if $u_1 = t$ then $u_2 = -it$ $\therefore$ $\boxed{u = t\begin{bmatrix} 1 \\ -i \end{bmatrix} \text{ for } \lambda = i}$

Find u with $Au = -iu$

$$\begin{matrix} -u_2 = -iu_1 \\ u_1 = -iu_2 \end{matrix} \implies u_2 = iu_1 \qquad \text{let } \begin{matrix} u_1 = t \\ u_2 = it \end{matrix} \implies \boxed{u = t\begin{bmatrix} 1 \\ i \end{bmatrix}}$$

Remark: these don't look like the answers in the back of the textbook ($\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -i \\ 1 \end{bmatrix}$). Convince yourself these are the same answers. (Key: $t \neq 0$ but t can be complex #.)

§9.5 #19 Find fundamental matrix for $x'(t) = Ax(t)$ for $A = \begin{bmatrix} -1 & 1 \\ 8 & 1 \end{bmatrix}$.

$$\begin{vmatrix} -1-\lambda & 1 \\ 8 & 1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-1) - 8 = \lambda^2 - 9 = (\lambda+3)(\lambda-3) = 0 \quad \therefore \quad \boxed{\lambda_1 = -3 \ \& \ \lambda_2 = 3}$$

$\lambda_2 = 3$: $\begin{bmatrix} -1 & 1 \\ 8 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3u \\ 3v \end{bmatrix} \implies \begin{matrix} -u + v = 3u \\ 8u + v = 3v \end{matrix} \implies v = 4u \implies u_2 = \begin{bmatrix} u \\ 4u \end{bmatrix} = u\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\lambda_1 = -3$: $\begin{bmatrix} -1 & 1 \\ 8 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -3u \\ -3v \end{bmatrix} \implies \begin{matrix} -u + v = -3u \\ 8u + v = -3v \end{matrix} \implies v = -2u \implies u_1 = \begin{bmatrix} u \\ -2u \end{bmatrix} = u\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

So we've found eigenvalue $\lambda_1 = -3$ with $u_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and eigenvalue $\lambda_2 = 3$ with $u_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ then we assemble the fundamental sol$^{\underline{n}}$'s:

$$x_1(t) = e^{-3t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$x_2(t) = e^{3t}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\longrightarrow \boxed{\Phi(t) = \left[x_1(t) \middle| x_2(t) \right] = \begin{bmatrix} e^{-3t} & e^{3t} \\ -2e^{-3t} & 4e^{3t} \end{bmatrix}}$$

the fundamental matrix is made of sol$^{\underline{n}}$'s.

$$A = \begin{bmatrix} 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \longrightarrow \begin{vmatrix} 4-\lambda & -1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & -3 \\ 0 & 0 & 1 & -2-\lambda \end{vmatrix} = (4-\lambda)(-\lambda)[(2-\lambda)(-2-\lambda)+3]$$

$$= \lambda(\lambda-4)[\lambda^2-1] = 0$$

$$\lambda_1 = 0,\ \lambda_2 = 4,\ \lambda_3 = 1,\ \lambda_4 = -1$$

Tedious calculation will reveal that

$$u_1 = \begin{pmatrix} 0.2425 \\ 0.9701 \\ 0 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ -0.9487 \\ -0.3162 \end{pmatrix} \quad u_4 = \begin{pmatrix} 0 \\ 0 \\ 0.7071 \\ 0.7071 \end{pmatrix}$$

for $\lambda_1 = 0$ for $\lambda_2 = 4$ for $\lambda_3 = 1$ for $\lambda_4 = -1$

These were calculated by my TI-89. Notice these all have length one, this is the convention of the calculator.

$$\Phi(t) = \begin{bmatrix} 0.2425 & e^{4t} & 0 & 0 \\ 0.9701 & 0 & 0 & 0 \\ 0 & 0 & -0.9487e^{t} & 0.7071e^{-t} \\ 0 & 0 & -0.3162e^{t} & 0.7071e^{-t} \end{bmatrix} = [X_1 | X_2 | X_3 | X_4]$$

- Again the fundamental matrix is made by concatenating fundamental sol^{n}'s to $x' = Ax$. As we've discussed the idea of eigenvalues & vectors allows us to assemble all the fundamental sol^{n}'s, provided the matrix is not unusual (we'll need "generalized" eigenvectors for some weird but important cases later on)

§9.5 # 26 Solve via matrix techniques the system

$$x' = 3x - 4y \quad\quad \rightarrow \begin{bmatrix} x \\ y \end{bmatrix}' = \underbrace{\begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \vec{x}' = A\vec{x}$$
$$y' = 4x - 7y$$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -4 \\ 4 & -7-\lambda \end{vmatrix} = (\lambda-3)(\lambda+7) + 16$$
$$= \lambda^2 + 4\lambda - 21 + 16$$
$$= (\lambda+5)(\lambda-1) = 0 \Rightarrow \underline{\lambda_1 = 1 \;\&\; \lambda_2 = -5}$$

Let $u_1 = \begin{pmatrix} u \\ v \end{pmatrix}$ and suppose $Au_1 = u_1$ then

$$\begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \rightarrow 3u - 4v = u \Rightarrow u = 2v \Rightarrow \boxed{u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

Let $u_2 = \begin{pmatrix} u \\ v \end{pmatrix}$ and suppose $Au_2 = -5u_2$

$$\begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -5u \\ -5v \end{bmatrix} \rightarrow 3u - 4v = -5u \Rightarrow 8u = 4v \Rightarrow v = 2u \Rightarrow \boxed{u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

Remark: the form of the eigenvectors I have chosen is pretty much arbitrairy. I just thought choosing $u = 1$ for u_1 or $v = 1$ for u_2 made it easy. In the end we are going to sum these over arbitrairy constants so it doesn't matter, we could just as correctly use $u_1 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$.

We find the sol^{n} using what we've learned,

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

In components,

$$\boxed{\begin{aligned} x(t) &= 2c_1 e^{t} + c_2 e^{-5t} \\ y(t) &= c_1 e^{t} + 2c_2 e^{-5t} \end{aligned}}$$

$$x'(t) = \underbrace{\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}}_{A} x(t) \quad \text{with } x(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 - 9 = \lambda^2 - 2\lambda + 1 - 9 = \lambda^2 - 2\lambda - 8 = (\lambda-4)(\lambda+2) = 0$$

$$\underline{\lambda_1 = 4,\ \lambda_2 = -2}$$

$$\lambda_1 = 4 \quad \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 4u \\ 4v \end{bmatrix} \rightarrow \begin{matrix} u + 3v = 4u \\ u = v \end{matrix} \rightarrow u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -2 \quad \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -2u \\ -2v \end{bmatrix} \rightarrow \begin{matrix} u + 3v = -2u \\ u = -v \end{matrix} \rightarrow u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus the general sol^{n} is $x(t) = c_1 e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

We should fit the intial data (alternatively we could use the idea of # 26 of §9.4, but we'll not do that here)

$$x(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} c_1 + c_2 &= 3 \\ + \quad c_1 - c_2 &= 1 \\ \hline 2c_1 &= 4 \end{aligned} \Rightarrow \boxed{c_1 = 2} \quad \& \quad c_2 = c_1 - 1 = \boxed{1 = c_2}.$$

$$\therefore \boxed{x(t) = 2e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

you can check, $x(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ as it should.

§9.5 # 41 Consider $ay'' + by' + cy = 0$ for constants a, b, c.
Change this 2^{nd} order DEq^n to two first order DEq^n's by

$$x_1 = y$$
$$x_2 = y'$$

In terms of x_1 & x_2 the DEq^n becomes

$$x_1' = x_2$$
$$x_2' = -\frac{b}{a}y' - \frac{c}{a}y = -\frac{b}{a}x_2 - \frac{c}{a}x_1$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longleftrightarrow \frac{dx}{dt} = Ax$$

So we've recast the constant coeff. 2^{nd} order ODE as a system of 2 1^{st} order ODEs in normal form.
Now we'll show the characteristic eq^n & auxillarly equations match (sometimes I call the aux. eq^n the characteristic eq^n, so it's good they're the same.)

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -c/a & -\lambda - b/a \end{vmatrix}$$

$$= \lambda(\lambda + b/a) + c/a$$
$$= \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0$$
$$\Rightarrow \boxed{a\lambda^2 + b\lambda + c = 0}$$

(§9.6 #2) $A = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$ find sol^{n} of system $X'(t) = AX(t)$.

$$\begin{vmatrix} -\lambda-2 & -5 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda+2)(\lambda-2)+5 = \lambda^2+1 = 0 \quad \therefore \boxed{\lambda = \pm i} \quad \begin{array}{l} \nearrow \lambda_1 = i \\ \searrow \lambda_2 = -i \end{array}$$

Eigenvectors are found as usual,

$$\begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} iu \\ iv \end{bmatrix} \Rightarrow \begin{array}{c} u+2v = iv \\ u = (i-2)v \end{array} \Rightarrow U_1 = \begin{bmatrix} i-2 \\ 1 \end{bmatrix}$$

$$U_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a + ib \qquad \begin{array}{l} Re(U_1) = a \\ Im(U_1) = b \end{array}$$

Well not quite as usual, <u>because</u> we don't need to work as much to find U_2 in fact it'd be $U_2 = a - ib$.

$$AU_1 = iU_1 \Rightarrow AU_1^* = -iU_1^* \quad \& \quad AU_2 = -iU_2$$

(taking the complex conjugate)

identify $U_2 = U_1^*$ will work $U_2 = (a+ib)^* = a - ib$.

- Anyway we don't need to find U_2 anyway. Instead we assemble the sol^{n} from a and b as described on p. 546 and eq^{n}'s (6) & (7)

$$\boxed{X(t) = c_1\left(\cos t\begin{bmatrix} -2 \\ 1 \end{bmatrix} - \sin t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + c_2\left(\sin t\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \cos t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)}$$

(§9.6 #6) $A = \begin{bmatrix} -2 & -2 \\ 4 & 2 \end{bmatrix}$

$$\begin{vmatrix} -2-\lambda & -2 \\ 4 & 2-\lambda \end{vmatrix} = (\lambda-2)(\lambda+2) + 8 = \lambda^2+4 = 0 \quad \therefore \boxed{\lambda = \pm 2i}$$

$$\begin{bmatrix} -2 & -2 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2iu \\ 2iv \end{bmatrix} \rightarrow \quad 4u + 2v = 2iv$$

$$4u = (2i-2)v$$

$$u = \tfrac{1}{2}(i-1)v$$

$$u_1 = \begin{bmatrix} i-1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a + ib$$

$$X_1(t) = \cos 2t \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \sin 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$X_2(t) = \sin 2t \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \cos 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boxed{\Phi(t) = [X_1(t) \mid X_2(t)] = \begin{bmatrix} -\cos 2t - \sin 2t & -\sin 2t + \cos 2t \\ 2\cos 2t & 2\sin 2t \end{bmatrix}}$$

(§9.6 #13) $x'(t) = \begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix} x(t)$

$$\begin{vmatrix} -3-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda+3) + 2 = \lambda^2 + 4\lambda + 5 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i = \alpha \pm i\beta \quad \begin{pmatrix} \alpha = -2 \\ \beta = 2 \end{pmatrix}$$

$$\begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = (-2+i)\begin{bmatrix} u \\ v \end{bmatrix} \longrightarrow \quad 2u - v = (i-2)v$$

$$2u = (i-1)v$$

$$u = \tfrac{1}{2}(i-1)v$$

$$u_1 = \begin{bmatrix} i-1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a+ib$$

$$X_1(t) = e^{-2t}\left(\cos t \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$X_2(t) = e^{-2t}\left(\sin t \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

§9.6 #13 The general sol^{n} is thus (continuing from last pg.)

$$X(t) = c_1 X_1 + c_2 X_2 = c_1 e^{-2t}\begin{bmatrix} -\cos t - \sin t \\ 2\cos t \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} -\sin t + \cos t \\ 2\sin t \end{bmatrix}$$

a.) Suppose $X(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ find sol^{n}.

$$X(0) = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$-c_1 + c_2 = -1 \quad \Rightarrow \quad c_2 = -1$$

$$2c_1 = 0 \quad \Rightarrow \quad c_1 = 0$$

$$\boxed{X(t) = e^{-2t}\begin{bmatrix} \sin t - \cos t \\ -2\sin t \end{bmatrix}}$$

(§9.7 #1) Solve $x'(t) = Ax(t) + f(t)$ where $A = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix}$ and $f(t) = \begin{bmatrix} -11 \\ -5 \end{bmatrix}$

We find the homogeneous sol$^{\underline{n}}$ to begin,

$$\begin{vmatrix} 6-\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} = (\lambda-3)(\lambda-6) - 4$$
$$= \lambda^2 - 9\lambda + 14$$
$$= (\lambda-2)(\lambda-7) = 0 \Rightarrow \underline{\lambda_1 = 2 \ \& \ \lambda_2 = 7}$$

$$\begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2u \\ 2v \end{bmatrix} \rightarrow \begin{matrix} 6u + v = 2u \\ v = -4u \end{matrix} \rightarrow u_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 7u \\ 7v \end{bmatrix} \rightarrow \begin{matrix} 6u + v = 7u \\ v = u \end{matrix} \rightarrow u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

} eigenvectors for A

$$x_h(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ -4 \end{bmatrix} + c_2 e^{7t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$x_h(t)$ solves corresponding homog. eq$^{\underline{n}}$ $x_h'(t) = Ax_h(t)$.

Guess $x_p(t) = \begin{bmatrix} a \\ b \end{bmatrix}$ then $x_p'(t) = 0$ so substituting yield

$$\begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} -11 \\ -5 \end{bmatrix} = \mathbf{0}$$

need to determine the values of a & b to make this eq$^{\underline{n}}$ true.

I'll compute the inverse for a change of pace,

$$\begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix}^{-1} = \frac{1}{18-4}\begin{bmatrix} 3 & -1 \\ -4 & 6 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 3 & -1 \\ -4 & 6 \end{bmatrix} = A^{-1}$$

Consider then,

$$Ax_p + f = 0 \Rightarrow Ax_p = -f$$
$$\Rightarrow x_p = A^{-1}(-f) \quad \text{since } A^{-1}A = I \text{ and } Ix_p = x_p$$
$$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 3 & -1 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 11 \\ 5 \end{bmatrix}$$
$$= \frac{1}{14}\begin{bmatrix} 33-5 \\ -44+30 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Thus the general sol$^{\underline{n}}$ is simply $x = x_h + x_p$ →

$$\boxed{x(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ -4 \end{bmatrix} + c_2 e^{7t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}}$$

(§9.7 #4) Solve $x'(t) = Ax + f$ where $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ & $f(t) = \begin{bmatrix} -4\cos t \\ -\sin t \end{bmatrix}$

To begin find homogeneous sol$^{\underline{n}}$,

$$\begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = (\lambda-2)^2 - 4 = \lambda^2 - 4\lambda + 4 - 4 = \lambda(\lambda-4) = 0$$

$$\underline{\lambda_1 = 0 \text{ \& } \lambda_2 = 4}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0 \Rightarrow \begin{matrix} 2u + 2v = 0 \\ -u = v \end{matrix} \Rightarrow u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 4u \\ 4v \end{bmatrix} \Rightarrow \begin{matrix} 2u + 2v = 4u \\ 2v = 2u \end{matrix} \Rightarrow u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus $x_h(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Now guess $x_p = a\cos t + b\sin t$ where a & b are yet to be determined constant vectors.

$x_p' = -a\sin t + b\cos t$

Substituting into the DEq$^{\underline{n}}$ yields,

$$-a\sin t + b\cos t = A(a\cos t + b\sin t) + \cos t\begin{bmatrix} -4 \\ 0 \end{bmatrix} + \sin t\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Now $\sin t$ & $\cos t$ are Linearly independent so they have independent coefficients, we can read off two matrix eq^{n}'s from the above

$$\sin t: \; -a = Ab + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\cos t: \; b = Aa + \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

} we need to somehow uncouple these eq^{n}'s so that either a or b is eliminated.

💡 $a = -Ab + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so just stick this into ↑ to get

$$b = A\left(-Ab + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

$$\Rightarrow (A^2 + I)b = A\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

$$\underbrace{\left(\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)}_{\left(\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)} b = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

§ 9.7 #4 Continued — We're still working on determining a and b. We just elimated a thru substitution and then reduced the problem of finding b to solving the eq^n,

$$\begin{bmatrix} 9 & 8 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Could solve this many ways, I'll use Kramer's Rule this time (last time I used multiplication by inverse in #1)

$$b_1 = \frac{\begin{vmatrix} -2 & 8 \\ 2 & 9 \end{vmatrix}}{\begin{vmatrix} 9 & 8 \\ 8 & 9 \end{vmatrix}} = \frac{-18-16}{81-64} = \frac{-34}{17} = -2$$

$$b_2 = \frac{\begin{vmatrix} 9 & -2 \\ 8 & 2 \end{vmatrix}}{\begin{vmatrix} 9 & 8 \\ 8 & 9 \end{vmatrix}} = \frac{18+16}{17} = \frac{34}{17} = 2$$

$$\Bigg\} \quad b = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

hmm. it's an eigenvector of A in fact. Just an acident I think.

Now we can use this to calculate a,.

$$a = -Ab + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= -\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= 0 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \therefore \; a = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then $x = x_h + x_p$ as usual so,

$$X(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin t \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Remark: When we did undetermined coeff. for $aY''+bY'+cY = g(t)$ it was always more trouble when $g(t) = \sin(t)$ or $\cos(t)$ as opposed to $g(t) = 6$. Here that trouble is amplified.

§9.7 #5 Set up the guess for $x_p(t)$ but dont calculate $a, b, \ldots$
For $x' = Ax + f$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ & $f(t) = e^{-2t}\begin{bmatrix} t \\ 3 \end{bmatrix}$

- To make the correct guess must insure no overlap with x_h (which means we've gotta find it.)

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (\lambda-1)(\lambda-2) = 0 \quad \therefore \lambda_1 = 1 \ \& \ \lambda_2 = 2$$

This will lead us to $x_h = c_1 e^{t} u_1 + c_2 e^{2t} u_2$
however we've got e^{-2t} for $f(t)$, no overlap.

Notice $f(t) = e^{-2t}\begin{bmatrix} 0 \\ 3 \end{bmatrix} + te^{-2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ this suggests

$$\boxed{x_p(t) = ae^{-2t} + bte^{-2t}.}$$

<u>Remark</u>: #7, #9 also have no overlap, you can check.

§9.7#7 $f(t) = \begin{bmatrix} \sin 3t \\ t \end{bmatrix}$ take derivatives and you'll find $\cos(3t)$ and constants, but that's it. Thus,

$$\boxed{x_p(t) = a\sin 3t + b\cos 3t + ct + d}$$

(a, b, c, d:) constant [2×1] matrices a.k.a 2-component column vectors.

§9.7#9 $f(t) = \begin{bmatrix} e^{2t} \\ e^{3t} \end{bmatrix} = e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

it is clear that derivatives of $f(t)$ will just be similar terms, always a e^{2t} or e^{3t} in every term,

$$\boxed{x_p(t) = ae^{2t} + be^{3t}}$$

<u>Remark</u>: I'm using the logic that the naive (no overlap) particular Sol$^{\underline{n}}$ is formed by taking a linear combination of the functions that appear in the list

$\{f(t), f'(t), f''(t), \ldots\}$ this must be finite list to use undetermined coeff. technique.

§9.7 #11 Solve $x'(t) = Ax(t) + f(t)$ by variation of parameters for systems of DEq^{n}'s. Here let us suppose

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad f(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Find the fundamental sol^{n}'s.

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \quad \therefore \quad \lambda = \pm i$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} iu \\ iv \end{bmatrix} \rightarrow v = iu \rightarrow u_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix} = a + ib$$

As discussed in § 9.6 this gives fund. sol^{n}'s,

$$X_1 = \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} = X_1(t)$$

$$X_2 = \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} = X_2(t)$$

Thus the fundamental matrix is found,

$$\Phi(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

And the inverse is calculated, notice $\det(\Phi(t)) = \cos^2 t + \sin^2 t = 1$.

$$\Phi^{-1}(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

Then formula (11) says the sol^{n} is formed by,

$$\begin{aligned} x(t) &= \Phi(t)c + \Phi(t)\int \Phi^{-1}(t) f(t)\, dt \\ &= \Phi c + \Phi \int \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} dt \\ &= \Phi c + \Phi \int \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} dt \\ &= \Phi c + \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} \\ &= \Phi c + \begin{bmatrix} \sin t \cos t - \sin t \cos t \\ -\sin^2 t - \cos^2 t \end{bmatrix} \\ &= \boxed{c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = x(t)} \end{aligned}$$

Remark: this is easier than undetermined coefficients, the tables have turned.

§9.7 #16 Solve $x'(t) = Ax + f(t)$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $f(t) = \begin{bmatrix} 8\sin t \\ 0 \end{bmatrix}$

From #11 where we had the same A we recall

$$\Phi(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \quad \& \quad \Phi^{-1}(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

Let use variation of parameters to find sol^n $x(t)$

$$\begin{aligned}
x(t) &= \Phi c + \Phi \int \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 8\sin t \\ 0 \end{bmatrix} dt \\
&= \Phi c + \Phi \int \begin{bmatrix} 8\sin t \cos t \\ 8\sin^2 t \end{bmatrix} dt \\
&= \Phi c + \Phi \begin{bmatrix} -4\cos^2 t \\ -4(\sin t \cos t - t) \end{bmatrix} \\
&= \Phi c + \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} 4\cos^2 t \\ 4(-\sin t\cos t + t) \end{bmatrix} \\
&= \Phi c + \begin{bmatrix} 4t\sin t - 4\cos t \\ 4t\cos t \end{bmatrix} \\
&= \boxed{c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + 4t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} - 4 \begin{bmatrix} \cos t \\ 0 \end{bmatrix} = x(t)}
\end{aligned}$$

Remark: If we had tried undetermined coefficients here we would have needed to modify the naive guess because $f(t)$ overlaps. We can see this in retrospect because of the "t" appearing in $x_p(t)$, but the "t" is not motivated purely from $f(t) = \begin{bmatrix} 8\sin t \\ 0 \end{bmatrix} \underset{\text{naively}}{\Longrightarrow} x_p = (\sin t)a + (\cos t)b$

§ 9.7 #21 Solve the following

$$X'(t) = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} X(t) + \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$$

to begin we'll find the fundamental matrix Φ.

$$\begin{vmatrix} -\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = \lambda(\lambda-3)+2 = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2) = 0 \quad \underline{\lambda_1 = 1,\ \lambda_2 = 2}$$

$$\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow 2v = u \Rightarrow u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow X_1 = e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2u \\ 2v \end{bmatrix} \Rightarrow 2v = 2u \Rightarrow u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow X_2 = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus we find $\Phi(t) = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}$ note $\det(\Phi(t)) = e^{3t}$

$$\Phi^{-1}(t) = \frac{1}{e^{3t}} \begin{bmatrix} e^{2t} & -e^{2t} \\ -e^t & 2e^t \end{bmatrix} = \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix}$$

Now recall we can fit $X(0) = x_0$ if we use (13) of pg. 554,

a.) let $X(0) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ then

$$\begin{aligned}
X(t) &= \Phi(t)\Phi^{-1}(0)x_0 + \Phi(t)\int_0^t \Phi^{-1}(s)\begin{bmatrix} e^s \\ -e^s \end{bmatrix} ds \\
&= \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} + \Phi(t)\int_0^t \begin{bmatrix} e^{-s} & -e^{-s} \\ -e^{-2s} & 2e^{-2s} \end{bmatrix}\begin{bmatrix} e^s \\ -e^s \end{bmatrix} ds \\
&= \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \Phi(t)\int_0^t \begin{bmatrix} 2 \\ -3e^{-s} \end{bmatrix} \\
&= \begin{bmatrix} 2e^t + 3e^{2t} \\ e^t + 3e^{2t} \end{bmatrix} + \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}\begin{bmatrix} 2t \\ -3 + 3e^{-t} \end{bmatrix} \\
&= \begin{bmatrix} 2e^t + 3e^{2t} \\ e^t + 3e^{2t} \end{bmatrix} + \begin{bmatrix} 4te^t + e^{2t}(-3+3e^{-t}) \\ 2te^t + e^{2t}(-3+3e^{-t}) \end{bmatrix} \\
&= \begin{bmatrix} 2e^t + \cancel{3e^{2t}} + 4te^t - \cancel{3e^{2t}} + 3e^t \\ e^t + \cancel{3e^{2t}} + 2te^t - \cancel{3e^{2t}} + 3e^t \end{bmatrix} \\
&= \boxed{\begin{bmatrix} 5e^t + 4te^t \\ 4e^t + 2te^t \end{bmatrix} = X(t)}
\end{aligned}$$

§9.7 #4 revisited, use variation of parameters this time, refer to our previous work we can write,

$$\Phi(t) = \begin{bmatrix} 1 & e^{4t} \\ -1 & e^{4t} \end{bmatrix} \qquad \det(\Phi(t)) = e^{4t} + e^{4t} = 2e^{4t}$$

$$\Phi^{-1}(t) = \frac{1}{\det \Phi(t)} \begin{bmatrix} e^{4t} & -e^{4t} \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ e^{-4t} & e^{-4t} \end{bmatrix}$$

Then calculate the sol^{n} by (11),

$$\begin{aligned}
X(t) &= \Phi c + \Phi \int \Phi^{-1}(t) f(t)\, dt \\
&= \Phi c + \Phi \int \frac{1}{2}\begin{bmatrix} 1 & -1 \\ e^{-4t} & e^{-4t} \end{bmatrix}\begin{bmatrix} -4\cos t \\ -\sin t \end{bmatrix} dt \\
&= \Phi c + \Phi \int \frac{1}{2}\begin{bmatrix} -4\cos t + \sin t \\ e^{-4t}(-4\cos t - \sin t) \end{bmatrix} dt \\
&= \Phi c + \frac{1}{2}\Phi \begin{bmatrix} \int(-4\cos t + \sin t)\, dt \\ \int e^{-4t}(-4\cos t - \sin t)\, dt \end{bmatrix} \\
&= \Phi c + \frac{1}{2}\Phi \begin{bmatrix} -4\sin t - \cos t \\ e^{-4t}\cos t \end{bmatrix}
\end{aligned}$$

2^{nd} components integration not necessarily easy.

$$\begin{aligned}
&= \Phi c + \frac{1}{2}\begin{bmatrix} 1 & e^{4t} \\ -1 & e^{4t} \end{bmatrix}\begin{bmatrix} -4\sin t - \cos t \\ e^{-4t}\cos t \end{bmatrix} \\
&= \Phi c + \frac{1}{2}\begin{bmatrix} -4\sin t - \cos t + \cos t \\ +4\sin t + \cos t + \cos t \end{bmatrix}
\end{aligned}$$

it's neat how the exponentials cancel out.

$$\begin{aligned}
&= \Phi c \quad \frac{1}{2}\begin{bmatrix} -4\sin t \\ 4\sin t \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 2\cos t \end{bmatrix} \\
&= \boxed{c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sin t\begin{pmatrix} -2 \\ 2 \end{pmatrix} + \cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix} = X(t)}
\end{aligned}$$

<u>Remark</u>: this problem is a little difficult any which way we do it, but it's nice to see we were correct before (or wrong twice, but that would be amazing)

(§9.8 #1) $A = \begin{bmatrix} 3 & -2 \\ 0 & 3 \end{bmatrix}$ Compute e^{At}. To begin I'll do it the hard way.

$$A^2 = \begin{bmatrix} 3 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 9 & -12 \\ 0 & 9 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 9 & -12 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 27 & -54 \\ 0 & 27 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 81 & -216 \\ 0 & 81 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 243 & -810 \\ 0 & 243 \end{bmatrix}$$

Then
$$e^{tA} = I + tA + \tfrac{1}{2}t^2A^2 + \cdots$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 9 & -12 \\ 0 & 9 \end{bmatrix} + \tfrac{1}{2}t^2\begin{bmatrix} 27 & -54 \\ 0 & 27 \end{bmatrix} + \tfrac{1}{3!}t^3\begin{bmatrix} 81 & -216 \\ 0 & 81 \end{bmatrix} + \cdots$$

It's not obvious to me how to complete this calculation with direct calculation. Now lets do it as §9.8 suggests.

$$\begin{vmatrix} 3-\lambda & -2 \\ 0 & 3-\lambda \end{vmatrix} = (\lambda-3)^2 = 0 \quad \therefore \quad \lambda = 3 \text{ with multiplicity } 2$$

Find eigenvector u_1; $(A-3I)u_1 = 0$

$$\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow -2v = 0 \longrightarrow u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Next the generalized eigenvector u_2; $(A-3I)^2 u_2 = 0$

$$\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow 0 \cdot u_2 = 0, \text{ choose } u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(need to pick a vector linearly indep. from u_1.)

Then assemble sol^{n}'s,

$$X_1(t) = e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$X_2(t) = e^{3t}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = e^{3t}\begin{bmatrix} -2t \\ 1 \end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} e^{3t} & -2te^{3t} \\ 0 & e^{3t} \end{bmatrix} \longrightarrow \Phi^{-1}(t) = \frac{1}{e^{9t}}\begin{bmatrix} e^{3t} & 2te^{3t} \\ 0 & e^{3t} \end{bmatrix}$$

From the fundamental matrix we calculate

$$e^{At} = \Phi(t)\Phi^{-1}(0) = \begin{bmatrix} e^{3t} & -2te^{3t} \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{e^{3t}\begin{bmatrix} 1 & -2t \\ 0 & 1 \end{bmatrix} = e^{tA}}$$

§9.8 #1 Lets try to follow directions this time,

$P(\lambda) = (\lambda-3)^2$ characteristic polynomial

$P(A) = (A-3)^2 = 0$ Cayley Hamilton Th^{m} matrix makes characteristic polynomial zero.

Then notice

$$\begin{aligned} e^{At} &= e^{3It}e^{(A-3I)t} \\ &= e^{3It}\left(I + (A-3I)t + \tfrac{1}{2}(A-3I)^2t^2 + \cdots\right) \quad (A-3I)^2 \to 0 \text{ by observation above.} \\ &= e^{3It}\left(I + (A-3I)t\right), \quad \text{note } A-3I = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \text{ thus} \\ &= \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 1 & -2t \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{3t} & -2te^{3t} \\ 0 & e^{3t} \end{bmatrix} \\ &= \boxed{e^{3t}\begin{bmatrix} 1 & -2t \\ 0 & 1 \end{bmatrix} = e^{At}} \end{aligned}$$

neat, we've computed the same answer in very seemingly different ways.

§9.8 #7 $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Recall from §9.7 #11

$$\Phi(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \quad \& \quad \Phi^{-1}(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$$e^{tA} = \Phi(t)\Phi^{-1}(0) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix} = \boxed{\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} = e^{tA}}$$

This one we could even calculate directly if you don't mind some power series arguments,

$$A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow A^2 = -I$$

$$A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A \Rightarrow A^3 = -A$$

$$e^{tA} = \sum_{n=0}^{\infty}\frac{(tA)^{2n}}{(2n)!} + \sum_{m=0}^{\infty}\frac{(tA)^{2m+1}}{(2m+1)!} \quad \left(\text{Just splitting } \sum_{k=0}^{\infty}\frac{(tA)^k}{k!} \text{ into even \& odd powers}\right)$$

$$= \sum_{n=0}^{\infty}\frac{t^{2n}}{2n!}(A^2)^n + \sum_{m=0}^{\infty}\frac{t^{2m+1}}{(2m+1)!}A(A^2)^m$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n)!}I + \sum_{m=0}^{\infty}(-1)^m\frac{t^{2m+1}}{(2m+1)!}A = (\cos t)I + \sin t\,A = e^{tA}$$

§4.8 #10 $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

Remark: it is interesting that here we have 3 eigenvectors, whereas in #17 we'll only get 2 eigenvectors

$$\begin{vmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & 2 \\ 2 & 2 & -\lambda \end{vmatrix} = -\lambda(\lambda^2-4) - 2(-2\lambda-4) + 2(4+2\lambda)$$
$$= (\lambda+2)[-\lambda(\lambda-2)+4+4]$$
$$= (\lambda+2)[-\lambda^2+2\lambda+8]$$
$$= -(\lambda+2)(\lambda^2-2\lambda-8)$$
$$= -(\lambda+2)(\lambda+2)(\lambda-4) = 0 \Rightarrow \begin{array}{ll} \lambda_1 = -2 & m_1 = 2 \\ \lambda_2 = 4 & m_2 = 1 \end{array}$$

multiplicities

- Find u_1 with $Au_1 = -2u_1 \Leftrightarrow (A+2I)u_1 = 0$

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow 2u + 2v + 2w = 0$$

Let $u = 1$ and $v = 0$ then $2 + 2w = 0 \Rightarrow w = -1 \therefore u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

- Next observe that $u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is also an eigenvector L.I. from u_1.

- Find u_3 with $Au_3 = 4u_3$ that is $(A-4I)u_3 = 0$

$$\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow -4u + 2v + 2w = 0$$

Let $\underline{u = 1}$ and $\underline{v = 1} \Rightarrow -4 + 2 + 2w = 0 \Rightarrow \underline{w = 1}$

$$u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Remark: my choices were guided by linear independence.

We can assemble the fundamental matrix,

$$\Phi(t) = \begin{bmatrix} e^{-2t} & e^{-2t} & e^{4t} \\ 0 & -e^{-2t} & e^{4t} \\ -e^{-2t} & 0 & e^{4t} \end{bmatrix} \qquad \Phi^{-1}(t) = \frac{1}{3}\begin{bmatrix} e^{2t} & e^{2t} & -2e^{2t} \\ e^{2t} & -2e^{2t} & e^{2t} \\ e^{-4t} & e^{-4t} & e^{-4t} \end{bmatrix}$$

$$e^{tA} = \Phi(t)\Phi^{-1}(0) = \frac{1}{3}\begin{bmatrix} e^{4t}+2e^{-2t} & e^{4t}-e^{-2t} & e^{4t}-e^{-2t} \\ e^{4t}-e^{-2t} & e^{4t}+2e^{-2t} & e^{4t}-e^{-2t} \\ e^{4t}-e^{-2t} & e^{4t}-e^{-2t} & e^{4t}+2e^{-2t} \end{bmatrix}$$

§9.8 #17 $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix}$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & -5 & -4-\lambda \end{vmatrix} = -\lambda(\lambda(\lambda+4)+5) - 1(+2)$$
$$= -\lambda(\lambda^2+4\lambda+5) - 2$$
$$= -\lambda^3 - 4\lambda^2 - 5\lambda - 2$$
$$= -(\lambda+1)^2(\lambda+2) = 0$$

Find eigenvector with $\lambda = -1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -u \\ -v \\ -w \end{bmatrix} \qquad \begin{aligned} v &= -u \\ w &= -v \\ -2u-5v-4w &= -w \end{aligned}$$

Let $u = 1$ then $v = -1$ and $w = 1$ note $-2+5-4 = -1 = w$ (consistent)

$$u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \boxed{X_1(t) = e^{-t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$$

Next the generalized eigenvector $(A+I)^2 u_2 = 0$.

$$(A+I)^2 u_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -5 & -3 \end{bmatrix}^2 \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -4 & -2 \\ 4 & 8 & 4 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$u + 2v + w = 0$$

Let $u = 0$ then let $v = 1$ so $2 + w = 0 \therefore \underline{w = -2}$.

$$u_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$ ← generalized eigenvector, it satisfies $(A+I)^2 u_2 = 0$

$$X_2(t) = e^{-t}\{u_2 + t(A+I)u_2\}$$
$$= e^{-t}\left\{\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + t\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -5 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}\right\}$$
$$= e^{-t}\left\{\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$$
$$= \boxed{\begin{bmatrix} te^{-t} \\ (1-t)e^{-t} \\ (-2+t)e^{-t} \end{bmatrix} = X_2(t)}$$

§9.8 #17 Continued, find eigenvector u_3 with $Au_3 = -2u_3$,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -2u \\ -2v \\ -2w \end{bmatrix} \longrightarrow \begin{array}{c} v = -2u \\ w = -2v \end{array}$$

Let $u = 1$ then $v = -2$ and thus $w = 4$ this checks against $-2u - 5v - 4w = -2 + 10 - 16 = -8 = -2(4) = -2w$.

$$u_3 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \longrightarrow \boxed{X_3(t) = e^{-2t}\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}}$$

Collecting our thoughts, the general sol$^{\underline{n}}$ is

$$\boxed{X(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} t \\ 1-t \\ t-2 \end{bmatrix} + c_3 e^{-2t}\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}}$$

§9.8 #5 $A = \begin{bmatrix} -2 & 0 & 0 \\ 4 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$ $|A - \lambda I| = \begin{vmatrix} -2-\lambda & 0 & 0 \\ 4 & -2-\lambda & 0 \\ 1 & 0 & -2-\lambda \end{vmatrix} = (-2-\lambda)^3 = -(\lambda+2)^3 = 0$

The characteristic polynomial is $P(x) = -(x+2)^3$.

$$e^{tA} = e^{-2It}\, e^{t(A+2I)}$$

$$= e^{-2It}\left(I + t(A+2I) + \tfrac{1}{2}t^2(A+2I)^2 + \tfrac{1}{3!}t^3(A+2I)^3 + \cdots\right)$$

By $P(A) = 0$ Cayley Ham. Th$^{\underline{m}}$. ($(A+2I)^3 \to 0$)

$$= e^{-2It}\left(I + t\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \tfrac{1}{2}t^2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \boxed{e^{-2t}\begin{bmatrix} 1 & 0 & 0 \\ 4t & 1 & 0 \\ t & 0 & 1 \end{bmatrix} = e^{tA}}$$

§9.8 #21 Use results of #5 to solve $X'(t) = AX(t)$ with $X(0) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Recall $X(t) = \Phi(t)\Phi^{-1}(0)X_0$ is a sol$^{\underline{n}}$ to $X'(t) = AX(t)$ with $X(0) = X_0$

Thus, using $e^{tA} = \Phi(t)\Phi^{-1}(0)$ we find

$$X(t) = e^{tA}X_0 = e^{-2t}\begin{bmatrix} 1 & 0 & 0 \\ 4t & 1 & 0 \\ t & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \boxed{e^{-2t}\begin{bmatrix} 1 \\ 4t+1 \\ t-1 \end{bmatrix} = X(t)}$$