

INTRODUCTION TO DIFFERENTIAL EQUATIONS

1

- (1) - table of contents
- (2) - introduction to notes
- (3-5) - basic terminology (§1.1-1.2)
- (5) - direction fields (§1.3)
- (6) - Euler's Method (§1.4)
- (6-7) - Separation of Variables (§2.2)
- (8-9) - integrating factors (§2.3)
- (10-12) - exact eqⁿ's (§2.4)
- (13-16) - applications & physics (§3.1-3.4)
- (17-21) - theory of linear ODEs (§4.2, 4.3, 6.2, 6.3)
- (22) - linear independence for fncts. (§4.2, 6.1)
- (23) - differential operators (6.3)
- (24-25) - Complex analysis (§4.3, 6.2)
- (26) - algebra review
- (27) - more differential ops. (§6.3)
- (28-30) - solⁿ to n^{th} order const. (§4.2, 4.3, 6.2, 6.3) coeff. linear ODE.
- (31-33) - undetermined coeff. (§4.4, 4.5, 6.3)
- (34) - annihilator method (§6.3)
- (34) - superposition principle (§4.5)
- (35-39) - Variation of parameters (§4.6, 6.4)
- (40-48) - Laplace Transforms (§7.1, 7.2, 7.3)
- (49-51) - Inverse Laplace Transforms (§7.4)
- (52-54) - Solving ODEs via Laplace. (§7.5)
- (55-61) - More Laplace Transforms (§7.6)
- (62-65) - fun with convolutions (§7.7)
- (66-69) - DIRAC DELTA FUNCTION (§7.8)
- (70-71) - Systems of ODEs, how (§7.9) to solve with Laplace Transforms.
- (72-74) - 2 eqⁿ's & 2 unknowns
- (75-82) - n eqⁿ's & n unknowns and (§9.1, 9.2, 9.3) how to use matrices to solve these algebraic problems.
- (83-84) - Normal Form (§9.4)
- (85-88) - theory of systems of linear (§9.4-9.5) diff. eqⁿ's in normal form.
- (85) - linear independence of (§9.5) vector-valued functions
- (89-94) - solⁿ to homogeneous systems (§9.5, 9.6) of linear ODEs (easy cases)
- (100-101) - undetermined coeff. for systems (§9.7)
- (102-103) - variation of parameters for (§9.7) systems of ODEs.
- (104-123) - matrix exponential function and (§9.8) how to use it to solve any linear ODE in normal form.

HOMEWORK SOLUTIONS

- (H1-H22) - first order DEqⁿ's, based on prerequisite knowledge and 3-16.
- (H23-H44) - higher order DEq's based on 17-39.
- (H45-H70) - Laplace Methods follows 40-69.
- (H71-H103) - Systems of Linear ODEs follows 70-123.

Remark: certain homeworks supercede my notes, but not generally. All the homework is from our text,

"FUNDAMENTALS OF DIFFERENTIAL EQUATIONS AND BOUNDARY VALUE PROBLEMS" 4th Ed. NAGLE, SAFF & SNIDER

I have indicated the sections most relevant to the notes parenthetically. If you wish to see more applications the text has much more than I've included in these notes. My main intent is to explain the math, I leave the motivation to you (the adult)

INTRODUCTION TO MY NOTES

(2)

These notes follow your text mostly. If I can teach you a fraction of the text I'll be happy, it's a great book. Anyway I tend to use some abbreviations here and there so let me list them here for your convenience,

Symbols	Meaning
§	section
$\exists$	there exists
$\nexists$	there does not exist
$a \in B$	a is an element of B
$\mathbb{N}$	Natural #'s 1, 2, 3, ...
$\mathbb{Z}$	Integers ..., -1, 0, 1, 2, ...
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if
iff	$\Leftrightarrow$
$\therefore$	therefore
//	change in topic
$\forall$	for all
$\equiv$	definition
def ⁿ	definition
Sol ⁿ	solution
w.r.t	with respect to
$\notin$	not an element of

Symbols	Meaning
f-la	formula
Th ^m	theorem
eq ⁿ	equation
s.t.	such that
$\mathbb{R}$	real numbers ($-\infty, \infty$)
α	alpha
β	beta
γ	gamma
δ	delta
θ	theta
Ω	omega
ω	omega
ψ	psi
ϕ	phi

LI = linear independence
 ODE = ordinary differential equation
 fct = function

BASIC TERMINOLOGY

Let us suppose that x is the independent variable and y is the dependent variable. That means we can view y as a function of x . An ordinary differential equation (O.D.E.) is an eqⁿ of the form,

$$F(y^{(n)}, y^{(n-1)}, \dots, y^{(2)}, y', y, x) = 0 \quad n^{\text{th}} \text{ order differential eq}^n$$

Where $y^{(n)} = \frac{d^n y}{dx^n}$ and $y' = \frac{dy}{dx}$. This highest derivative that appears gives the order of the DEqⁿ. It is called ordinary because all the derivatives are just w.r.t. x . When a differential eqⁿ has partial derivatives (like $\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t}$ for example) then it is called a partial differential eqⁿ (PDE). Those are more subtle to solve and we relegate the discussion of how to solve PDE's to Ma 401.

Defⁿ/ A n^{th} order linear differential eqⁿ has the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x) \frac{dy}{dx} + a_0(x) y = F(x)$$

Examples

$$\begin{aligned} \frac{dy}{dx} &= y^2 && \text{(non-linear // solved by separating variables)} \\ \frac{dy}{dx} + y &= \cos(x) && \text{(linear // solve with method of undeter. coefficients)} \\ y'' + 3y' + 3y &= \tan(x) && \text{(linear // variation of parameters will solve it)} \\ \left(\frac{dy}{dx}\right)^3 + \left(\frac{dy}{dx}\right)^2 &= \sin(x) && \text{(non-linear // if you can solve this I'll be impressed.)} \end{aligned}$$

Defⁿ/ A solution to a differential eqⁿ $F(y^{(n)}, \dots, x) = 0$ is a function f such that

$$F(f^{(n)}(x), f^{(n-1)}(x), \dots, f'(x), f(x), x) = 0$$

- Given a n^{th} order ODE $F(y^{(n)}, \dots, y', x) = 0$ we will find there is not a unique solⁿ just on the basis of the DEqⁿ itself. Roughly speaking solving a n^{th} order DEqⁿ amounts to taking n -integrals. For each integral we get an integration constant, so as those constants are arbitrary we see we get infinitely many solⁿ's. As a common abuse of terminology I will often refer to this infinite family of solutions as the "general solⁿ".
- Now, if DEqⁿ's are to describe the real world this is discouraging, it would seem we can never uniquely describe a system with an ODE. The solⁿ to this is simple, physical systems are in practice modelled by DEqⁿ's + INITIAL CONDITIONS.

Defⁿ/A solⁿ to an initial value problem (IVP) on an interval I containing x_0 is a solⁿ to a DEqⁿ $F(y^{(n)}, \dots, y', x) = 0$ that also satisfies n -extra initial conditions

$$y(x_0) = y_0, \quad \frac{dy}{dx}(x_0) = y_1, \quad \dots, \quad \frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1}$$

where $y_0, y_1, \dots, y_{n-1}$ are given constants.

- Usually an initial value problem has a unique solⁿ, but not always (you have a hwk. problem to illustrate this)

Th^m(1) Given an IVP $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$.

If f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle that contains the point (x_0, y_0) then the IVP has a unique solⁿ in some interval $(x_0 - \delta, x_0 + \delta)$ for $\delta > 0$.

- The solⁿ may or may not extend to the whole rectangle. This is a local theorem, it just tells us about solⁿ's near the point x_0 .

Remark: Sometimes we will be given an implicit formula to describe a function. For some examples rather than possessing an explicit f-lu for $f(x)$ we'll instead say the solⁿ is the locus of points satisfying some relation.

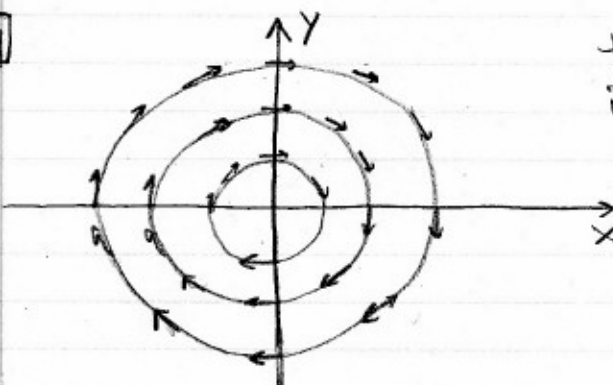
Example: $x^2 + y^2 = 1$ is a solⁿ to $\frac{dy}{dx} = -\frac{x}{y}$

At the level of functions we'd have two solⁿs $f_1(x) = \sqrt{1-x^2}$ for $y > 0$ and $f_2(x) = -\sqrt{1-x^2}$ for $y < 0$.

DIRECTION FIELDS

One method of visualizing a DEqⁿ is to draw its direction field.

[E1]

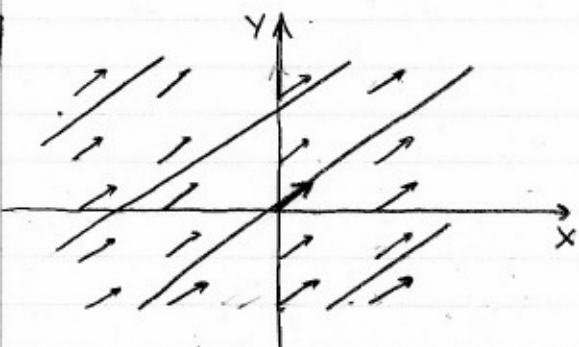


the little arrows $\rightarrow$ indicate the slope that tangents to the solⁿ curves must have. Here I've tried to sketch:

$$\frac{dy}{dx} = -\frac{x}{y}$$

and a few of the solⁿs which happen to be circles.

[E2]



here I've sketched the direction field for $\frac{dy}{dx} = 1$. Solⁿs have slope one everywhere, that means they're lines of slope one,

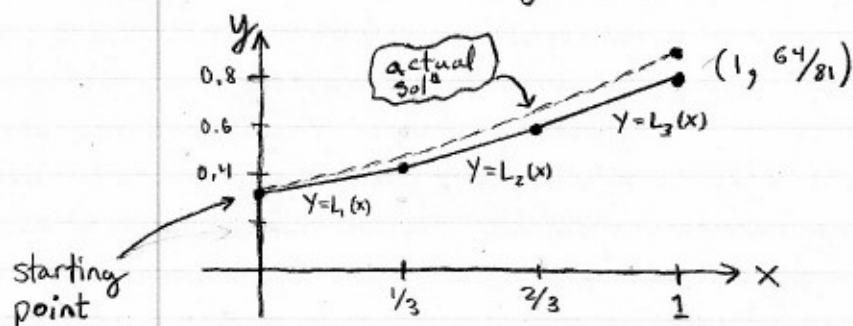
$$y = x + b$$

Remark: I cannot draw all the solⁿs, they literally fill the (xy) -plane*. We can see that DEqⁿs do not usually give unique solⁿs. (*ok, to be careful we should throw out $(0,0)$ in [E1])

Euler's Method

- Closely related to the idea of the direction field.
- In short, Euler's Method generates approximate solⁿ's to a DEqⁿ by replacing the real solⁿ with a piecewise linear version of the solⁿ.

E3 $\frac{dy}{dx} = y$ given the initial data $y(0) = \frac{1}{3}$ find $y(1)$ using Euler's method with 3-steps.



Euler's method with 3 Steps says $y(1) \approx \frac{64}{81}$

$$\begin{aligned} L_1(x) &= \frac{1}{3} + \left(\frac{1}{3}\right)(x-0) \Rightarrow L_1\left(\frac{1}{3}\right) = \frac{1}{3} + \frac{1}{9} = \frac{4}{9} = y_1 \\ L_2(x) &= \frac{4}{9} + \left(\frac{4}{9}\right)(x-\frac{1}{3}) \Rightarrow L_2\left(\frac{2}{3}\right) = \frac{4}{9} + \frac{4}{27} = \frac{16}{27} = y_2 \\ L_3(x) &= \frac{16}{27} + \left(\frac{16}{27}\right)(x-\frac{2}{3}) \Rightarrow L_3(1) = \frac{16}{27} + \frac{16}{81} = \frac{64}{81} = y_3 \end{aligned}$$

Remark: Euler's method & Direction fields require a lot of brute force arithmetic, as such they are best implemented by technology. We will focus on those methods which do not require numerical assistance. You should be aware that many problems defy closed form solⁿ's so we are driven to solve numerically.

Separation of Variables

E1 $\frac{dy}{dx} = y \Rightarrow \int \frac{dy}{y} = \int dx \Rightarrow \ln|y| = x + \tilde{C} \Rightarrow y = ce^x$

$y(0) = c = \frac{1}{3} \Rightarrow y = \frac{1}{3}e^x$ (exact solⁿ to **E3** above)

- let's compare the approximation to the real deal,

$y(1) = \frac{1}{3}e^1 = 0.9061$ compared to $\frac{64}{81} = 0.79$

E2 $\frac{dy}{dx} = \frac{6x^5 - 2x + 1}{\cos(y) + \sec^2(y)}$, find the general solⁿ implicitly

$$\int (\cos(y) + \sec^2(y)) dy = \int (6x^5 - 2x + 1) dx$$

$$\boxed{\sin(y) + \tan(y) = X^6 - X^2 + X + C} \leftarrow \text{implicit general sol}^n$$

E3 $Y' = X^3(1-Y)$ with $Y(0) = 3$, find explicit solⁿ to IVP.
Recall that $Y' = \frac{dY}{dx}$ when x is the independent variable,

$$\int \frac{dY}{1-Y} = \int x^3 dx \Rightarrow -\ln|1-Y| = \frac{1}{4} x^4 + C_1$$

$$\Rightarrow \ln|1-Y| = C_2 - x^4/4$$

$$\Rightarrow |1-Y| = e^{C_2 - x^4/4}$$

$$\Rightarrow 1-Y = \pm e^{C_2} e^{-x^4/4}$$

$$\Rightarrow \boxed{Y = 1 + ce^{-x^4/4}}$$

Now $Y(0) = 3 \Rightarrow 3 = 1 + C \therefore C = 2 \Rightarrow \boxed{Y = 1 + 2e^{-x^4/4}}$

E4 x = position
 $v = \frac{dx}{dt}$ = velocity
 $a = \frac{dv}{dt}$ = acceleration

Suppose that the acceleration is constant, in particular consider $a = g \in \mathbb{R}$. Find velocity as a function of time, and then position.

$$\frac{dv}{dt} = g \Rightarrow \int dv = \int g dt \Rightarrow \boxed{v = gt + C}$$

To find v as a function of x we could find $x(t)$ then solve for t , but we can get around finding $x(t)$ if we use the chain rule $\frac{dv}{dt} = \frac{dx}{dt} \frac{dv}{dx}$ but $\frac{dx}{dt} = v$ so,

$$v \frac{dv}{dx} = g \Rightarrow \int v dv = \int g dx \Rightarrow \frac{1}{2} v^2 = gx + C$$

$$\Rightarrow \boxed{v = \pm \sqrt{2gx + C}}$$