Introduction to Differential Equations

Homework Solutions

<u>remark</u>: certain homeworks supercede my notes, but not generally. All the homework is from our text,

"Fundamentals of Differential Equations and Boundary Value Problems" 4$^{\underline{th}}$ Ed, Nagle, Saff & Snider

I have indicated the sections most relevant to the notes parenthetically. If you wish to see more <u>applications</u> the text has <u>much</u> more than I've included in these notes. My main intent is to explain the math, I leave the motivation to you (the adult)

INTRODUCTION TO MY NOTES

These notes follow your text mostly. If I can teach you a fraction of the text I'll be happy, it's a great book. Anyway I tend to use some abbreviations here and there so let me list them here for your convenience,

Symbols	Meaning
§	section
$\exists$	there exists
$\nexists$	there does not exist
$a \in B$	a is an element of B
$\mathbb{N}$	Natural #'s 1, 2, 3, ...
$\mathbb{Z}$	Integers ..., -1, 0, 1, 2, ...
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if
iff	$\Leftrightarrow$
$\therefore$	therefore
$\#$	change in topic
$\forall$	for all
$\equiv$	definition
def^n	definition
Sol^n	solution
w.r.t	with respect to
$\notin$	not an element of

Symbols	Meaning
f-la	formula
Th^m	theorem
eq^n	equation
s.t.	such that
$\mathbb{R}$	real numbers $(-\infty, \infty)$
α	alpha
β	beta
γ	gamma
δ	delta
θ	theta
Ω	omega
ω	omega
ψ	psi
φ	phi

LI = linear independence
ODE = ordinary differential equation
fnct = function

Basic Terminology

Let us suppose that x is the independent variable and y is the dependent variable. That means we can view y as a function of x. An ordinary differential equation (O.D.E.) is an eq^{n} of the form,

$$\boxed{F(y^{(n)}, y^{(n-1)}, \dots, y^{(2)}, y', y, x) = 0 \qquad n^{th} \text{ order differential eq}^{n}}$$

Where $y^{(n)} = \frac{d^n y}{dx^n}$ and $y' = \frac{dy}{dx}$. This highest derivative that appears gives the order of the DEq^{n}. It is called ordinary because all the derivatives are just w.r.t. x. When a differential eq^{n} has partial derivatives (like $\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial \psi}{\partial t}$ for example) then it is called a partial differential eq^{n} (PDE). Those are more subtle to solve and we relagate the discussion of how to solve PDE's to Ma 401.

Def^{n}/ A n^{th} order linear differential eq^{n} has the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x)$$

Examples

$\frac{dy}{dx} = y^2$ (non-linear // solved by seperating variables)

$\frac{dy}{dx} + y = \cos(x)$ (linear // solve with method of undeter. coefficients)

$y'' + 3y' + 3y = \tan(t)$ (linear // variation of parameters will solve it)

$\left(\frac{dy}{dx}\right)^3 + \left(\frac{dy}{dx}\right)^2 = \sin(x)$ (non-linear // if you can solve this I'll be impressed.)

Def^{n}/ A solution to a differential eq^{n} $F(y^{(n)}, \dots, x) = 0$ is a function f such that

$$F(f^{(n)}(x), f^{(n-1)}(x), \dots, f'(x), f(x), x) = 0$$

- Given a n^{th} order ODE $F(y^{(n)}, \dots, y', x) = 0$ we will find there is not a unique sol^{n} just on the basis of the DEq^{n} itself. Roughly speaking solving a n^{th} order DEq^{n} amounts to taking n-integrals. For each integral we get an integration constant, so as those constants are arbitrary we see we get infinitely many sol^{n}'s. As a common abuse of terminology I will often refer to this infinite family of solutions as the "general sol^{n}".

- Now, if DEq^{n}'s are to describe the real world this is discouraging, it would seem we can never uniquely describe a system with an ODE. The sol^{n} to this is simple, physical systems are in practice modelled by <u>DEq^{n}'s</u> + <u>INTIAL CONDITIONS</u>.

Def^{n}/ A sol^{n} to an <u>intial value problem</u> (IVP) on an interval I containing x_o is a sol^{n} to a DEq^{n} $F(y^{(n)}, \dots, y', x) = 0$ that <u>also</u> satisfies n-extra intial conditions

$$y(x_o) = y_o \quad , \quad \frac{dy}{dx}(x_o) = y_1 \quad , \dots , \quad \frac{d^{n-1}y}{dx^{n-1}}(x_o) = y_{n-1}$$

where $y_o, y_1, \dots, y_{n-1}$ are given constants.

- Usually an intial value problem has a unique sol^{n}, but not always (you have a hwk. problem to illustrate this)

Th^{m}(1) Given an IVP $\frac{dy}{dx} = f(x,y)$, $y(x_o) = y_o$. If f and $\frac{\partial f}{\partial y}$ are continous in a rectangle that contains the point (x_o, y_o) then the IVP has a unique sol^{n} in some interval $(x_o - \delta, x_o + \delta)$ for $\delta > 0$.

- The sol^{n} may or may not extend to the whole rectangle. This is a local theorem, it just tells us about sol^{n}'s <u>near</u> the point x_o.

Remark: Sometimes we will given an implicit formula to describe a function. For some examples rather than possesing an explicit f-la for $f(x)$ we'll instead say the sol^n is the locus of points satisfying some relation.

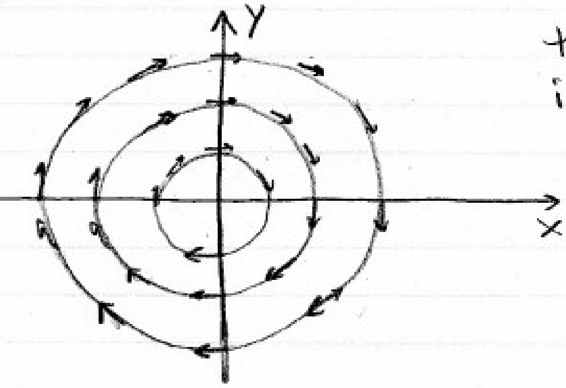
Example: $x^2 + y^2 = 1$ is a sol^n to $\frac{dy}{dx} = \frac{-x}{y}$

At the level of functions we'd have two sol^n's $f_1(x) = \sqrt{1-x^2}$ for $y > 0$ and $f_2(x) = -\sqrt{1-x^2}$ for $y < 0$.

Direction Fields

One method of visualizing a DEq^n is to draw it's direction field.

$\boxed{E1}$

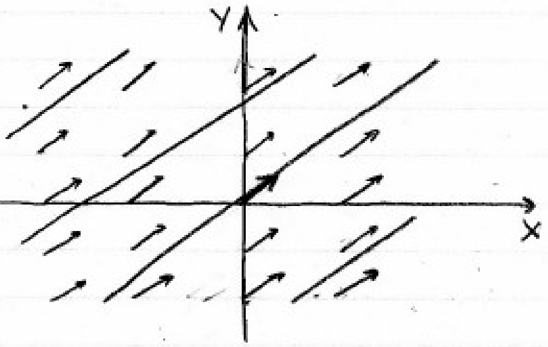


the little arrows → indicate the slope that tangents to the sol^n curves must have. Here I've tried to sketch:

$$\frac{dy}{dx} = \frac{-x}{y}$$

and a few of the sol^n's which happen to be circles.

$\boxed{E2}$



here I've sketched the direction field for $\frac{dy}{dx} = 1$. Sol^n's have slope one every where, that means they're lines of slope one,

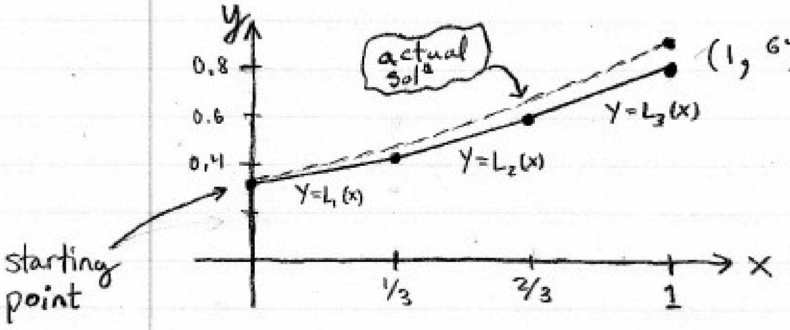
$$y = x + b$$

Remark: I cannot draw all the sol^n's, they literally fill the (xy)-plane.* We can see that DEq^n's do not usually give unique sol^n's. *(ok, to be careful we should throw out (0,0) in $\boxed{E1}$)

Euler's Method

- Closely related to the idea of the direction field.
- In short, Euler's Method generates approximate sol^n's to a DEq^n by replacing the real sol^n with a piecewise linear version of the sol^n.

$\boxed{E3}$ $\frac{dy}{dx} = y$ given the intial data $y(0) = 1/3$ find $y(1)$ using Euler's method with 3 - steps.



Eulers method with 3 Steps says $y(1) \cong \frac{64}{81}$

$$
\begin{aligned}
L_1(x) &= 1/3 + (1/3)(x-0) &&\Rightarrow L_1(1/3) = 1/3 + 1/9 = 4/9 = y_1 \\
L_2(x) &= 4/9 + (4/9)(x - 1/3) &&\Rightarrow L_2(2/3) = 4/9 + 4/27 = 16/27 = y_2 \\
L_3(x) &= 16/27 + (16/27)(x - 2/3) &&\Rightarrow L_3(1) = 16/27 + 16/81 = 64/81 = y_3
\end{aligned}
$$

Remark: Eulers method & Direction fields require alot of brute force arthimatic, as such they are best implemented by technology. We will focus on those methods which do not require numerical assistance. You should be aware that many problems defy closed form sol^n's so we are driven to solve numerically.

Separation of Variables

$\boxed{E1}$
$$\frac{dy}{dx} = y \Rightarrow \int \frac{dy}{y} = \int dx \Rightarrow \ln|y| = x + \tilde{c} \Rightarrow y = ce^x$$

$$y(0) = c = 1/3 \Rightarrow y = \frac{1}{3}e^x$$ (exact sol^n to $\boxed{E3}$ above)

- lets compare the approximation to the real deal,

$$y(1) = \frac{1}{3}e^1 = 0.9061 \quad \text{compared to} \quad \frac{64}{81} = 0.79$$

E2 $\frac{dy}{dx} = \frac{6x^5 - 2x + 1}{\cos(y) + \sec^2(y)}$, find the general sol^n implicitly

$$\int (\cos(y) + \sec^2(y))\,dy = \int (6x^5 - 2x + 1)\,dx$$

$$\boxed{\sin(y) + \tan(y) = x^6 - x^2 + x + c} \leftarrow \text{implicit general sol}^n$$

E3 $y' = x^3(1-y)$ with $y(0) = 3$, find explicit sol^n to IVP.
Recall that $y' = \frac{dy}{dx}$ when x is the independent variable,

$$\int \frac{dy}{1-y} = \int x^3\,dx \Rightarrow -\ln|1-y| = \frac{1}{4}x^4 + c_1$$

$$\Rightarrow \ln|1-y| = c_2 - x^4/4$$

$$\Rightarrow |1-y| = e^{c_2 - x^4/4}$$

$$\Rightarrow 1-y = \pm e^{c_2} e^{-x^4/4}$$

$$\Rightarrow \boxed{y = 1 + ce^{-x^4/4}}$$

Now $y(0) = 3 \Rightarrow 3 = 1 + c \quad \therefore \quad c = 2 \Rightarrow \boxed{y = 1 + 2e^{-x^4/4}}$

E4 x = position
$v = \frac{dx}{dt}$ = velocity
$a = \frac{dv}{dt}$ = acceleration

Suppose that the acceleration is constant, in particular consider $a = g \in \mathbb{R}$. Find velocity as a function of time, and then position.

$$\frac{dv}{dt} = g \Rightarrow \int dv = \int g\,dt \Rightarrow \boxed{v = gt + c}$$

To find v as a function of x we could find $x(t)$ then solve for t, but we can get around finding $x(t)$ if we use the chain rule $\frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx}$ but $\frac{dx}{dt} = v$ so,

$$v\frac{dv}{dx} = g \Rightarrow \int v\,dv = \int g\,dx \Rightarrow \frac{1}{2}v^2 = gx + c$$

$$\Rightarrow \boxed{v = \pm\sqrt{2gx + c}}$$