Inverse Laplace Transforms

Def^n/ Given $F(s)$, if there is a function $f(t)$ continuous on $[0,\infty)$ with
$$\mathcal{L}\{f\} = F$$
then we say $f(t)$ is the <u>inverse Laplace transform of F</u>. We denote $f = \mathcal{L}^{-1}\{F\}$.

<u>Remark</u>: there are many possible choices for f given some particular F. This is due to fact that $\int_0^\infty e^{-st} f_1(t)\,dt = \int_0^\infty e^{-st} f_2(t)\,dt$ provided $f_1(t)$ & $f_2(t)$ disagree only at a few points. The result of the inverse transform is unique if we require f to be continuous. This is a subtle point & I've already said to much here...

E1 $\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}(t) = \boxed{t^2}$ since $\mathcal{L}\{t^2\}(s) = \frac{2}{s^3}$.

E2 $\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}(t) = \boxed{\sin(3t)}$ since $\mathcal{L}\{\sin(3t)\}(s) = \frac{3}{s^2+9}$

E3 $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-2s+5}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\}(t)$ ← Completed square in the denominator.

$\therefore$ $\boxed{f(t) = e^t\cos(2t)}$ as $\mathcal{L}\{f\}(s) = \frac{s-1}{(s-1)^2+4}$.

<u>Reminder</u>: to complete the square we simply want to <u>rewrite</u> a quadratic from $ax^2+bx+c \longrightarrow (x-h)^2+k$. To do this we just take $\frac{1}{2}$ of coefficient of x and then $\left(x+\frac{b}{2}\right)^2 = x^2+bx+\frac{b^2}{4}$ so we then have to subtract $b^2/4$ to be fair,
$$x^2+bx+c = \left(x+\frac{b}{2}\right)^2 - \frac{b^2}{4} + c.$$
It's easier for specific examples, as it stands we are getting dangerously close to deriving the quadratic formula.
$$x^2+2x+5 = (x+1)^2-1+5 = (x+1)^2+4$$
$$x^2+6x+5 = (x+3)^2-9+5 = (x+3)^2-4$$
In practice I just make sure the LHS & RHS are equal, you don't need to remember some algorithm really.

Th$^{\underline{m}}$(7) Inverse Laplace Transform is Linear provided we choose continous $f(t)$

$$\mathcal{L}^{-1}\{F+G\} = \mathcal{L}^{-1}\{F\} + \mathcal{L}^{-1}\{G\}$$

$$\mathcal{L}^{-1}\{cF\} = c\,\mathcal{L}^{-1}\{F\}$$

<u>Proof:</u> follows from Linearity of $\mathcal{L}$.

$$\mathcal{L}\{\mathcal{L}^{-1}\{F\}+\mathcal{L}^{-1}\{G\}\} = \mathcal{L}\{\mathcal{L}^{-1}\{F\}\} + \mathcal{L}\{\mathcal{L}^{-1}\{G\}\}$$

$$= F + G$$

Then $\mathcal{L}^{-1}\{F\}+\mathcal{L}^{-1}\{G\} = \mathcal{L}^{-1}\{F+G\}$, using $\mathcal{L}^{-1}\mathcal{L} = 1$ which is true as we choose $\mathcal{L}^{-1}\{F\}$ to be continous. The fact $c \in \mathbb{R}$ pulls out follows similarly.

E4 Find $f(t) = \mathcal{L}^{-1}\{F\}(t)$ for $F(s) = \dfrac{3s+2}{s^2+2s+10}$

notice $s^2+2s+10 = (s+1)^2+9 \Rightarrow e^{-t}\cos 3t$ & $e^{-t}\sin 3t$ in the answer. Lets work it out,

$$\frac{3s+2}{s^2+2s+10} = \frac{3s+2}{(s+1)^2+9} \quad : \text{completing square}$$

$$= \frac{3(s+1)}{(s+1)^2+9} + \frac{-3+2}{(s+1)^2+9}$$

want to make function of (s+1) then I had to subtract 3 since I added 3.

$$= 3\frac{s+1}{(s+1)^2+3^2} - \frac{1}{3}\frac{3}{(s+1)^2+3^2}$$

In view of the algebra above it should be obvious that

$$\mathcal{L}^{-1}\{F\}(t) = \boxed{3e^{-t}\cos(3t) - \frac{1}{3}e^{-t}\sin(3t) = f(t)}$$

E5 Consider $F(s) = \dfrac{s}{s^2+5s+6}$ find $\mathcal{L}^{-1}\{F\}(t) = f(t)$

Notice $\dfrac{s}{s^2+5s+6} = \dfrac{s}{(s+3)(s+2)} = \dfrac{A}{s+3} + \dfrac{B}{s+2}$ (recall <u>partial fractions</u>)

$$\Rightarrow s = A(s+2) + B(s+3) \quad \xrightarrow{s=-2} -2 = B \to B=-2$$

$$\xrightarrow{s=-3} -3 = -A \to A = 3$$

Thus we deduce

$$\mathcal{L}^{-1}\{F\}(t) = \mathcal{L}^{-1}\left\{\frac{3}{s+3} - \frac{2}{s+2}\right\}(t)$$

$$= 3\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}(t) - 2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t) = \boxed{3e^{-3t} - 2e^{-2t} = f(t)}$$

PARTIAL FRACTIONS

• See pgs. 121-122 for a few more examples from Calculus II.

So we have discussed how polynomials split into linear and irred. quadratic factors. This means if we have a rational function which is $\frac{p(s)}{q(s)}$ then $p(s)$ & $q(s)$ will factor, we assume $\deg(p) < \deg(q)$ for convenience (otherwise we'd do long division). In short, partial fractions says you can split up a rational function into a sum of "basic" rational functions. For "basic" rational functions we can readily see how to take the inverse transform. Partial fractions involves a number of cases as you may read in the text, but it is important to realize it is nothing more than undoing making a common denominator. I'll leave you with a few examples,

$$\frac{s^3-3}{(s+1)^3(s^2+1)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3} + \frac{Ds+E}{s^2+1}$$

$$\frac{s+3}{s^2(s-2)(s^2+3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} + \frac{Ds+E}{s^2+3} + \frac{Fs+G}{(s^2+3)^2}\ *$$

It is a simple, but tedious, matter to calculate the constants $A, B, C, \dots, G$ in the above. Notice on the RHS almost all the terms are easily inverse transformed. The term $*$ is subtle, just like in integration theory.

<u>Remark</u>: it is <u>crucial</u> to understand the difference between (s^2+1) and $(s+1)^2$. Note how different the inverse transforms of $\left(\frac{1}{s^2+1}\right)$ verses $\left(\frac{1}{(s+1)^2}\right)$

USING LAPLACE TRANSFORMS TO SOLVE IVPs:

$\boxed{E1}$ $\quad y'' - 2y' + 5y = 0 \qquad$ with $y(0) = 2$ & $y'(0) = 12$

Take the Laplace transform,

$$s^2 Y - s\,y(0) - y'(0) - 2(sY - y(0)) + 5Y = 0$$

Lets solve for Y,

$$(s^2 - 2s + 5)Y = 2s + 8$$

$$Y(s) = \frac{2s+8}{s^2 - 2s + 5}$$

We need to determine $y(t)$, so all we really need is to find $\mathcal{L}^{-1}\{Y\}(t)$. To do that we need to determine if $s^2 - 2s + 5$ will factor, note $b^2 - 4ac = 4 - 20 = -16 < 0$ thus it is an irreducible quad $\Rightarrow$ complete square then break into sine & cosine like pieces.

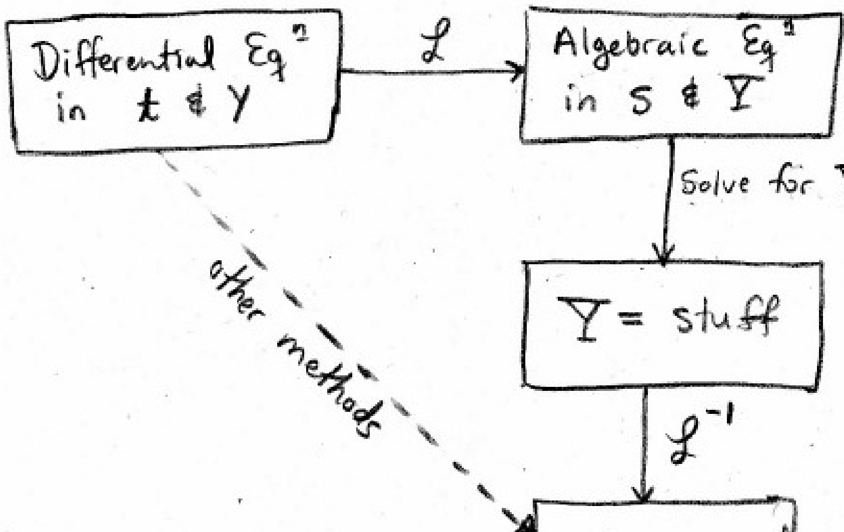
$$\frac{2s+8}{s^2-2s+5} = \frac{2s+8}{(s-1)^2+4}$$

$$= \underbrace{\frac{2(s-1)}{(s-1)^2+2^2}}_{\text{cosine}} + \underbrace{\frac{2}{2}\,\frac{8+2}{(s-1)^2+2^2}}_{\text{sine}}$$

Thus

$$\mathcal{L}^{-1}\{Y\}(t) = \boxed{2e^t\cos(2t) + 5e^t\sin(2t) = y(t)}$$

Pictorial Summary



• notice this method automatically encodes the intial conditions.

$\boxed{E2}$ Solve the repeated root problem $y'' + 4y' + 4y = 0$ subject to the intial conditions $y(0) = 1$ and $y'(0) = 1$. Taking Laplace transform yields,

$$s^2\mathcal{Y} - sy(0) - y'(0) + 4(s\mathcal{Y} - y(0)) + 4\mathcal{Y} = 0$$

$$(s^2+4s+4)\mathcal{Y} = s+5$$

$$\mathcal{Y} = \frac{s+5}{s^2+4s+4} = \frac{A}{s+2} + \frac{B}{(s+2)^2}$$

$$s+5 = A(s+2) + B$$

$$\underline{\text{Const}|} \quad 5 = 2A + B$$

$$\underline{s\,|} \quad \underline{1 = A} \quad \Rightarrow \quad B = 5 - 2A = \underline{3 = B}$$

$$\therefore \quad y(t) = \mathcal{L}^{-1}\{\mathcal{Y}\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2}\right\}(t)$$

$$\Rightarrow \boxed{y(t) = e^{-2t} + 3te^{-2t}}$$

Remark: The method of Laplace transforms has derived the curious te^{-2t} term. Before we just pulled it out of thin-air and argued that it worked. In defense of our earlier methods, the Laplace machine is not that intuitive either. At least we have one derivation now. Another route to explain the "t" in the double root sol^n is to use "reduction of order". I'll show you if your interested, it is §4.4 of the 3rd ed. of Nagel SAFF and SNIDER. We'll find another derivation of the "t" in chapter 9.

E3 Solve $w''(t) - 2w'(t) + 5w(t) = -8e^{\pi - t}$ given $w(\pi)=2$, $w'(\pi)=12$.
We need conditions at $t=0$ so to remedy being given them at π we introduce

$$Y(t) \equiv w(t+\pi) \Rightarrow \begin{array}{l} Y(0) = w(\pi) = 2 \\ Y'(0) = w'(\pi) = 12 \end{array}$$

Then,

$$w''(t+\pi) - 2w'(t+\pi) + 5w(t+\pi) = -8e^{\pi-(t+\pi)} = -8e^{-t}$$

Thus

$$Y'' - 2Y' + 5Y = -8e^{-t} \quad \text{with } Y(0)=2 \;\&\; Y'(0)=12.$$

Taking Laplace transform yields

$$(s^2-2s+5)\mathcal{Y} - 2s - 12 - 2(-2) = \frac{-8}{s+1}$$

$$\mathcal{Y} = \left(8 + 2s - \frac{8}{s+1}\right)\frac{1}{s^2-2s+5}$$

$$= \frac{3(s-1) + 2(4)}{(s-1)^2 + 2^2} - \frac{1}{s+1}$$

partial fractions ½ page of algebra here.

$$\underline{\mathcal{L}^{-1}\{\mathcal{Y}\}(t) = 3e^{t}\cos(2t) + 4e^{t}\sin(2t) - e^{-t} = Y(t)}$$

Then returning to our original problem note $w(t) = Y(t-\pi)$.

$$w(t) = 3e^{t-\pi}\cos(2(t-\pi)) + 4e^{t-\pi}\sin(2(t-\pi)) - e^{-t+\pi}$$

$$= 3e^{t-\pi}\cos(2t-2\pi) + 4e^{t-\pi}\sin(2t-2\pi) - e^{\pi-t}$$

$$= \boxed{3e^{t-\pi}\cos(2t) + 4e^{t-\pi}\sin(2t) - e^{\pi-t} = w(t)}$$

<u>Remark</u>: this example is important in that it show us how to use Laplace transforms to treat problems where the data is given at any time, not just zero as the formalism is set-up for.