n- eq^n's and n- unknowns and matrix notation

Let a_{ij}, b_i be constants and x_i be variables where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$ then a linear system of n algebraic eq^n is.

$$\boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n = b_n \end{array}}$$

each eq^n can be viewed as a dot-product

$$[a_{11}, a_{12}, \ldots, a_{1n}] \cdot [x_1, x_2, \ldots, x_n] = b_1 \quad : 1^{st} \text{ Eq}^n$$

$$[a_{i1}, a_{i2}, \ldots, a_{in}] \cdot [x_1, x_2, \ldots, x_n] = b_i \quad : i^{th} \text{ Eq}^n.$$

In matrix notation,

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{b}$$

The system of n- scalar eq^n's becomes the single matrix eq^n

$$\boxed{Ax = b}$$

Remark: I assume you've had some exposure to matrices before, if you haven't then it'd be wise to read the text before reading my notes, and please ask if you don't know what I mean by something here, it's likely just notation.

Matrix Multiplication

Given an matrix $A = (a_{ik})$ with m-rows and p-columns and another matrix $B = (B_{kj})$ with p-rows and n-columns we can define the product $AB = ((AB)_{ij})$ with m-rows and n-columns as follows:

$$\boxed{(AB)_{ij} = \sum_{k=1}^{p} A_{ik} B_{kj}}$$

This tells us what the (ij)-th slot in AB must be in terms of the components of A & B. Lets be more pictorial,

$$AB = \begin{bmatrix} \text{---} & A_1 & \text{---} \\ \text{---} & A_2 & \text{---} \\ & \vdots & \\ \text{---} & A_n & \text{---} \end{bmatrix} \begin{bmatrix} | & | & & | \\ B_1 & B_2 & \cdots & B_n \\ | & | & & | \end{bmatrix}$$

A_i = ith row of A

B_j = jth column of B

$$= \begin{bmatrix} \vec{A}_1 \cdot \vec{B}_1 & \vec{A}_1 \cdot \vec{B}_2 & \cdots & \vec{A}_1 \cdot \vec{B}_n \\ \vec{A}_2 \cdot \vec{B}_1 & \vec{A}_2 \cdot \vec{B}_2 & & \vec{A}_2 \cdot \vec{B}_n \\ \vdots & & & \vdots \\ \vec{A}_n \cdot \vec{B}_1 & \vec{A}_n \cdot \vec{B}_2 & \cdots & \vec{A}_n \cdot \vec{B}_n \end{bmatrix}$$

$\vec{A}_i \cdot \vec{B}_j$ the dot-product of $\vec{A}_i$ with $\vec{B}_j$ thinking of them as n-dim'l vectors.

Which shows,

$$\begin{aligned} (AB)_{ij} &= \vec{A}_i \cdot \vec{B}_j \\ &= \sum_{k=1}^{p} (\vec{A}_i)_k (\vec{B}_j)_k \\ &= \sum_{k=1}^{p} A_{ik} B_{jk} \end{aligned}$$

$(\vec{A}_i)_k = A_{ik}$

$(\vec{B}_j)_k = B_{jk}$

i = row index in A

j = column index in B

thus k has to range over columns and rows respectively.

E4

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} \langle 1,2 \rangle \cdot \langle 5,6 \rangle \\ \langle 3,4 \rangle \cdot \langle 5,6 \rangle \end{bmatrix} = \begin{bmatrix} 5+12 \\ 15+24 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$$

We see a (2×2) multiplied by (2×1) gives $(2\times\cancel{2})(\cancel{2}\times1) = (2\times1)$.

E5 Here we'll see a $(4\times 2)(2\times 2) = (4\times 2)$,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} \langle 1,2\rangle\cdot\langle 9,11\rangle & \langle 1,2\rangle\cdot\langle 10,12\rangle \\ \langle 3,4\rangle\cdot\langle 9,11\rangle & \langle 3,4\rangle\,\langle 10,12\rangle \\ \langle 5,6\rangle\cdot\langle 9,11\rangle & \langle 5,6\rangle\,\langle 10,12\rangle \\ \langle 7,8\rangle\cdot\langle 9,11\rangle & \langle 7,8\rangle\,\langle 10,12\rangle \end{bmatrix}$$

I'm just writing the dot-products for your benefit, normally I would not include this step.

$$= \begin{bmatrix} 9+22 & 10+24 \\ 27+44 & 30+48 \\ 45+66 & 50+72 \\ 63+88 & 70+96 \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 34 \\ 71 & 78 \\ 111 & 122 \\ 151 & 166 \end{bmatrix}$$

Def^{n}/ The <u>transpose</u> of a matrix $A = (A_{ij})$ is denoted A^T and is defined by $(A^T)_{ij} \equiv A_{ji}$. It is obtained by flipping rows to columns. A square matrix A with $A = A^T$ is called <u>symmetric</u>

E6 Let $v = [1\ 2\ 3]$ then $v^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$vv^T = [1\ 2\ 3]\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1+4+9 = 14$$

$$v^Tv = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}[1\ 2\ 3] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

- We see that matrix multiplication is <u>not</u> commutative $v^Tv \neq vv^T$, in fact they're not even the same size! Remember $A = B$ iff $A_{ij} = B_{ij}$ for all i and j, equality in a matrix eq^{n} $\approx$ equality in a bunch of ordinary eq^{n}'s.

- We also observe that $vv^T = \vec{v}\cdot\vec{v}$, so matrix multiplication can express the dot-product, suprise, surprise...

Algebra and Calculus of Matrices

Basically, it is very similar to the rules we used for vectors and vector-valued functions in ma 242. We do everything component-wise, equality, scalar multiplication addition of matrices, differentiation of matrix-valued function and integration of matrix-valued functions. A matrix-valued function of a real variable t is simply an assignment of a matrix $A(t)$ for each t, in fact this is a whole bunch of functions $a_{ij} : \mathbb{R} \to \mathbb{R}$.

$$\boxed{\begin{aligned} (A+B)_{ij} &\equiv a_{ij} + b_{ij} \\ (cA)_{ij} &\equiv c\,a_{ij} \\ (-A)_{ij} &\equiv -a_{ij} \\ \left(\frac{dA}{dt}\right)_{ij} &\equiv \frac{d}{dt}(a_{ij}) \\ \left(\int A\,dt\right)_{ij} &\equiv \int a_{ij}\,dt \end{aligned}}$$

$A = (a_{ij})$ and $B = (b_{ij})$, $c \in \mathbb{C}$.

$A = (a_{ij})$ and we assume the differentiations and integrations of the components make sense.

- I'll let you explore addition & multiplication as well as multiplication of matrices in the homework. It's not hard, but you need to do it for yourself.

$\boxed{E7}$ Let $A = \begin{bmatrix} 1 & 2t \\ 3t^2 & 4t^3 \end{bmatrix}$ then we can calculate,

$$3A = \begin{bmatrix} 3 & 6t \\ 6t^2 & 12t^3 \end{bmatrix} \qquad \int A\,dt = \begin{bmatrix} t+c_1 & t^2+c_2 \\ t^3+c_3 & t^4+c_4 \end{bmatrix} = \begin{bmatrix} t & t^2 \\ t^3 & t^4 \end{bmatrix} + C$$

$$\frac{dA}{dt} = \begin{bmatrix} 0 & 6 \\ 12t & 36t^2 \end{bmatrix} \qquad \int_0^2 A\,dt = \begin{bmatrix} 2-0 & 4-0 \\ 8-0 & 16-0 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 16 \end{bmatrix}$$

Properties of Matrix Algebra

Let A, B, C be matrices and α a number,

$$(AB)C = A(BC)$$
$$(A+B)+C = A+(B+C)$$
$$(A+B)C = AC + BC$$
$$A(B+C) = AB + AC$$
$$(\alpha A)B = \alpha(AB) = A(\alpha B)$$
$$A + 0 = A$$
$$A - A = 0$$
$$A I_n = I_m A = A$$

where $I_m = m\times m$ identity matrix
$I_n = n \times n$ identity matrix
$A = m \times n$ matrix

$$I_m = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Defn/ Let A be a square matrix (say $n \times n$) then the inverse matrix of A is another square matrix A^{-1} satisfying $A^{-1}A = I_n$.

Remark: when $A^{-1}A = I_n$ it follows that $AA^{-1} = I_n$ automatically. This in contrast to functional inverses where we had to check that $f\circ f^{-1}$ and $f^{-1}\circ f$ were both the identity function.

Notice that for a (2×2) matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$ the inverse is,

$$\boxed{A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.}$$

This is a nice formula to know, I've used it many times.

$$AA^{-1} = \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided $ad - bc \neq 0$

$$= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & -ab+ab \\ cd-dc & -bc+da \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(It's easy to check we're correct here)

<u>Remark</u>: we saw A^{-1} is not hard to find in 2×2 case. In general we use Gaussian Elimination on an the following matrix to find it. Best to see by example,

$\boxed{E8}$

$$A = \begin{bmatrix} 1 & 2 & 8 \\ 1 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 8 & 1 & 0 & 0 \\ 1 & 0 & 6 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \quad \text{start with } [A \mid I]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 8 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \quad R2 \rightarrow R2 - R1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 0 & 1 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \quad R1 \rightarrow R1 + R2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & -2 & 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R1 \rightarrow R1 - 3R3 \\ R2 \rightarrow R2 + R3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \quad \begin{array}{l} R2 \rightarrow -\tfrac{1}{2}(R2) \\ R3 \rightarrow \tfrac{1}{2}(R3) \end{array}$$

goal is to do row-operations until we change $[A \mid I] \rightarrow [I \mid A^{-1}]$

end with A^{-1}

$$\boxed{A^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 2 & -6 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$$

Let's Check our Answer,

$$AA^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 2 & 8 \\ 1 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 2 & -6 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 2-2 & -6-2+8 \\ 0 & 2 & -6+6 \\ 0 & 0 & 2 \end{bmatrix}$$

Thus $AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$ as req^d.

<u>Remark</u>: I do expect you to learn how to do row operations as above, however I'll allow you a calculator to check your work. These calculations take some practice & patience for most of us.

$Th^{\underline{m}}$ (1) Let A be an $(n \times n)$ matrix. The following statements are equivalent

(a.) A has no inverse; that defines A being "singular."

(b.) $\det(A) = 0$

(c.) $Ax = 0$ has nontrivial sol$^{\underline{n}}$'s $(x \neq 0)$

(d.) The columns (rows) of A form a linearly dependent set

Proof: take a linear algebra course, perhaps ma 405.

Remark: this $Th^{\underline{m}}$ is important to our discussion of eigenvectors, but it begs the question what about when $\det(A) \neq 0$? What happens then? Lets add a $Th^{\underline{m}}$

$Th^{\underline{m}}$ (Nonsingular Case) Let A be an $n \times n$ matrix. The following statements are equivalent

a.) A has an inverse (it is "non singular")

$AA^{-1} = I$

b.) $\det(A) \neq 0$

c.) $Ax = 0 \iff x = 0$ (only trivial sol$^{\underline{n}}$)

d.) The columns (rows) of A form LI set.

e.) $Ax = b$ with $x = [x_1, x_2, \ldots, x_n]^T$ has sol$^{\underline{n}}$ given by Kramer's Rule Assuming $A = [A_1 | A_2 | \cdots | A_n]$

$$x_1 = \frac{\det[b | A_2 | \cdots | A_n]}{\det(A)}$$

$$x_2 = \frac{\det[A_1 | b | A_3 | \cdots | A_n]}{\det(A)}$$

$$x_n = \frac{\det[A_1 | A_2 | \cdots | A_{n-1} | b]}{\det(A)}$$

Remark: there is much more to learn about matrices, especially their connection to linear operators... but our time is short ☹

More Calculus of Matrices

$$\boxed{\begin{aligned} &\frac{d}{dt}(CA) = C\frac{dA}{dt} \quad : \text{where } \frac{dC}{dt} = 0 \text{ that is } C \text{ is a constant matrix.} \\ &\frac{d}{dt}(A+B) = \frac{dA}{dt} + \frac{dB}{dt} \\ &\frac{d}{dt}(AB) = \frac{dA}{dt}B + A\frac{dB}{dt} \end{aligned}}$$

The formulas above are matrix eq^{n}'s. Each of them follows from the corresponding rule in ordinary calculus. I'll prove the interesting one. Let $A = (a_{ik})$ and $B = (b_{kj})$ where a_{ik} and b_{kj} are differentiable functions.

$$\begin{aligned}
\left(\frac{d}{dt}(AB)\right)_{ij} &\equiv \frac{d}{dt}(AB)_{ij} && : \text{diff. component-wise} \\
&= \frac{d}{dt}\left(\sum_{k=1}^{p} a_{ik} b_{kj}\right) && : \text{def}^{\underline{n}} \text{ of matrix multiplication.} \\
&= \sum_{k=1}^{p} \frac{d}{dt}(a_{ik} b_{kj}) && : \text{linearity of } \frac{d}{dt} \\
&= \sum_{k=1}^{p} \left(\frac{da_{ik}}{dt} b_{kj} + a_{ik}\frac{db_{kj}}{dt}\right) && : \text{product rule from ma 141.} \\
&= \sum_{k=1}^{p} \left(\frac{dA}{dt}\right)_{ik} b_{kj} + \sum_{k=1}^{p} a_{ik}\left(\frac{dB}{dt}\right)_{kj} \\
&= \left(\frac{dA}{dt}B + A\frac{dB}{dt}\right)_{ij}
\end{aligned}$$

Since this holds for each i & j we find equality

$$\frac{d}{dt}(AB) = \left(\frac{dA}{dt}\right)B + A\left(\frac{dB}{dt}\right).$$