THE MATRIX EXPONENTIAL FUNCTION

In the beginning of this course we solved $Y' = aY$ with the exponential sol^{n} $Y = e^{at}$. One might wonder if we can solve $x' = Ax$ with $x = e^{At}$. In fact we can, and implicitly we already have been with our eigenvector/value sol^{n}'s. It will take a few pages to clarify that claim. First we need to define and play with the <u>matrix exponential</u> e^A.

$$\boxed{\text{Def}^{n}/\quad e^A \equiv I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!}}.$$

this is motivated by the series expansion for the ordinary exponential function ($e^x = 1 + x + \frac{1}{2}x^2 + \cdots$)

$\boxed{E1}$ Let us take a special case $A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$. Such a matrix is called <u>diagonal</u>. Its easy to calculate,

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} \alpha^3 & 0 \\ 0 & \beta^3 \end{bmatrix} \quad \cdots \quad A^n = \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix}$$

Thus,

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}\begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{n=0}^{\infty} \frac{1}{n!}\alpha^n & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \end{bmatrix}$$

$$= \begin{bmatrix} e^{\alpha} & 0 \\ 0 & e^{\beta} \end{bmatrix}$$

The expontial of a diagonal matrix is simply a diagonal matrix with ordinary exponentials of the diagonal entries of the matrix. This works for $n \times n$ matrices just the same.

Th$^{\underline{m}}$ (7) Let A, B be $n \times n$ constant matrices and $t, s \in \mathbb{C}$,

1.) $e^{A0} = e^0 = I_n$

2.) $e^{A(t+s)} = e^{At}e^{As}$

3.) $(e^{At})^{-1} = e^{-At}$

4.) $e^{(A+B)t} = e^{At}e^{Bt}$ provided $AB = BA$.

5.) $e^{sI_nt} = e^{st}I_n$

Proof: Just use the def$^{\underline{n}}$ and calculate. The 5.) is just the same as $\boxed{E1}$ in special case,

$$e^{st}I_n = \begin{bmatrix} e^{st} & & & \\ & e^{st} & & \\ & & \ddots & \\ & & & e^{st} \end{bmatrix} = e^{\begin{bmatrix} st & & 0 \\ & st & \\ 0 & & \ddots\; st \end{bmatrix}}.$$

Remark: When $AB \neq BA$ 4.) is more complicated. If we define the "commutator of A and B" to be $[A, B] \equiv AB - BA$ we can calculate

$$e^A e^B = e^{A + B + \frac{1}{2}[A,B] + \frac{1}{12}[[A,B],A] - \frac{1}{12}[[B,A],B] + \cdots}$$

All the nested commutators vanish when $[A, B] = 0$. This is known as the Baker-Cambell-Hausdorff relation. I digress.

Proposition: $\frac{d}{dt}\left(e^{At}\right) = Ae^{At}$

Proof:
$$\begin{aligned}
\frac{d}{dt}\left(e^{At}\right) &= \frac{d}{dt}\sum_{n=0}^{\infty}\frac{(At)^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{d}{dt}(t^n)\frac{A^n}{n!} \\
&= \sum_{n=1}^{\infty}\frac{n}{n!}t^{n-1}A^n \qquad \text{since } n=0 \text{ term trivial} \\
&= \sum_{n=1}^{\infty}A\frac{(tA)^{n-1}}{(n-1)!} = A\sum_{k=0}^{\infty}\frac{(At)^k}{k!} = Ae^{At}.
\end{aligned}$$

$\boxed{\text{Th}^{\underline{m}}(8): \; e^{At} \text{ is a fundamental matrix for } \frac{dx}{dt} = Ax}$

Proof: Let $\Phi = e^{At}$ we wish to show $\Phi' = A\Phi$.

$$\frac{d}{dt}(\Phi) = \frac{d}{dt}(e^{At}) = Ae^{At} = A\Phi.$$

Now we also should show Φ is formed from n-LI Sol$^{\underline{n}}$'s $\Leftrightarrow W[x_1, x_2, \dots, x_n] = \det(\Phi) \neq 0$. We need to show $\det(e^{At}) \neq 0$. Now fortunately there is a beautiful identity involving this expression

$$\det(e^{At}) = e^{\text{trace}(A)}$$

Where $\text{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn}$ for $A = (a_{ij})$. The $\text{trace}(A) \in \mathbb{C}$ and in fact the ordinary exponential of a complex # is non-zero everywhere in the complex plane. Thus $\det(e^{At}) \neq 0$.

(I'll show you a proof of $\det(e^{At}) = e^{\text{trace}(A)}$ if you ask.)

$\boxed{\underline{\text{Remark}}: \; \frac{dx}{dt} = Ax \text{ has the general sol}^{\underline{n}} \; x = e^{At}c}$

Lets see before we had $x = c_1 e^{\lambda_1 t} u_1 + c_2 e^{\lambda_2 t} u_2 + \cdots + c_n e^{\lambda_n t} u_n$ where $\Phi = [e^{\lambda_1 t} u_1 | \cdots | e^{\lambda_n t} u_n]$ and $Au_j = \lambda_j u_j$ for $j = 1, 2, \dots, n$. How does this fit together with $\Phi = e^{At}$? Consider

$$\begin{aligned} e^{A} u_j &= (1 + A + \tfrac{1}{2}A^2 + \tfrac{1}{3!}A^3 + \cdots) u_j \\ &= (1 + \lambda_j + \tfrac{1}{2}\lambda_j^2 + \tfrac{1}{3!}\lambda_j^3 + \cdots) u_j \quad \text{applying } Au_j = \lambda_j u_j. \\ &= e^{\lambda_j} u_j \end{aligned}$$

Claim: if u is an eigenvector of A with eigenvalue λ then u is also an eigenvector of e^A with eigenvalue e^λ.

$$Au = \lambda u \quad \Rightarrow e^A u = e^\lambda u.$$

(we proved this on the last page.)

- Continuing to try to connect e^{At} verses our previous eigenvalue/vector sol$^{\underline{n}}$ consider,

$$\Phi = \left[e^{\lambda_1 t} u_1 \middle| \cdots \middle| e^{\lambda_n t} u_n\right] \quad \text{verses} \quad \tilde{\Phi} = e^{At}$$

Lets try multiplying $\tilde{\Phi}$ with $U = [u_1 | u_2 | \cdots | u_n]$

$$\begin{aligned} \tilde{\Phi} U &= e^{At} [u_1 | u_2 | \cdots | u_n] \\ &= \left[e^{\lambda_1 t} u_1 \middle| e^{\lambda_2 t} u_2 \middle| \cdots \middle| e^{\lambda_n t} u_n\right] \\ &= \Phi \end{aligned}$$

This means that $\Phi = [e^{\lambda_1 t} u_1 | \cdots | e^{\lambda_n t} u_n]$ is simply a reshuffling of e^{At} by multiplication by the constant matrix U.

Proposition: If Φ is a fundamental matrix for $x' = Ax$ then ΦB is a fundamental matrix for $x' = Ax$ provided B is a nonsingular matrix (B^{-1} exists or $\det(B) \neq 0$ equivalently.)

Proof:

$$\frac{d}{dt}(\Phi B) = \frac{d\Phi}{dt} B = A\Phi B \quad \text{and} \quad \underbrace{\det(\Phi B) = \det(\Phi)\det(B)}_{\text{nice property of determinants.}}$$

Thus $\det(\Phi B) \neq 0$ $\therefore$ the columns of ΦB form a LI set of sol$^{\underline{n}}$'s as needed.

Remark: one thing missing from our reasoning is the concept of a basis. We haven't enough time to do it justice, but it makes many issues clearer and aids calculation.

(THIS WHOLE PAGE IS A DIGRESSION, but I like it.)

Remark: If we choose n-LI eigenvectors of A that are mutually orthonormal as vectors; $u_i \cdot u_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ with $\det[u_1|u_2|\cdots|u_n] \neq 0$ and $Au_i = \lambda_i u_i$ then we can transform A into a diagonal matrix by the following trick,

$$\begin{aligned}
U^T A U &= U^T A [u_1|u_2|\cdots|u_n] \\
&= U^T [\lambda_1 u_1 | \lambda_2 u_2 | \cdots | \lambda_n u_n] \\
&= \begin{bmatrix} u_1^T \\ \hline u_2^T \\ \hline \vdots \\ \hline u_n^T \end{bmatrix} \left[\lambda_1 u_1 \middle| \lambda_2 u_2 \middle| \cdots \middle| \lambda_n u_n \right] \\
&= (u_i^T \lambda_j u_j) \qquad \text{note} \quad u_i \cdot u_j = u_i^T u_j = \delta_{ij}, \\
&= (\lambda_j \delta_{ij}) \\
&= \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} = D
\end{aligned}$$

This calculation shows that if A has n-LI eigenvectors then the corresponding Linear operator L $(LV = AV)$ can be represented by a diagonal matrix D in the eigenbasis $\{u_1, u_2, \ldots, u_n\}$. Obviously diagonal matrices are easier to work with than other forms So using an eigenbasis is nice (when A has enough vectors to give an eigenbasis, remember there are not necessarily n-LI eigenvectors for an arbitrary matrix.)

$\boxed{\text{Def}^{\underline{n}}/\ B \text{ is a nilpotent matrix if } B^k = 0 \text{ for some finite } k \in \mathbb{N}}$

We observe that the exponential terminates at order k for such a nilpotent matrix

$$e^B = I + B + \frac{1}{2}B^2 + \frac{1}{3!}B^3 + \cdots + \frac{1}{(k-1)!}B^{k-1}$$

Now suppose that some matrix A has characteristic polynomial $P(\lambda) = (\lambda - \alpha)^n$ then $P(A) = (A - \alpha I)^n = 0$ by Cayley Hamilton $Th^{\underline{m}}$. So think of $B = A - \alpha I$ it is nilpotent. Indeed observe,

$$\begin{aligned} e^{At} &= e^{(A - \alpha I + \alpha I)t} \\ &= e^{(A - \alpha I_n)t}\, e^{\alpha I_n t} \\ &= e^{\alpha t}\, e^{(A - \alpha I_n)t} \\ &= e^{\alpha t}\left\{ I + (A - \alpha I_n)t + \frac{1}{2}(A - \alpha I_n)^2 t^2 + \cdots + \frac{1}{(n-1)!}(A - \alpha I)^{n-1} t^{n-1} \right\} \end{aligned}$$

In this very special case we find e^{At} is not too difficult to calculate,

$\boxed{E2}$ Consider $x' = Ax$ where $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix}$

$$P(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ -2 & -2 & -1-\lambda \end{bmatrix} = -(\lambda - 1)^3 = 0$$

We thus know $(A - I)^3 = 0$. Then calculate,

$$A - I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix} \quad \$ \quad (A - I)^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore,

$$\begin{aligned} e^{At} &= e^t\left\{ I + (A - I)t \right\} \\ &= e^t \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & 0 \end{bmatrix} t \right\} \\ &= \begin{bmatrix} e^t(1+t) & te^t & te^t \\ e^t t & e^t(1+t) & te^t \\ -2te^t & -2te^t & e^t \end{bmatrix} \end{aligned}$$

<u>Remark</u>: we see that the matrix exponential will give more than just exponentials. It also yields $sol^{\underline{n}}$'s with te^t type terms. This is new.

Lemma (1) Let Φ and Ψ be fundamental matrices for $x' = Ax$. Then there exists a constant matrix C such that $\Phi(t) = \Psi(t)C$

Proof: Let $\Phi = [x_1|x_2|\cdots|x_n]$ and $\Psi = [y_1|y_2|\cdots|y_n]$ notice that each x_j can be written as linear combination of y_i's.

$$x_j = C_{1j}y_1 + C_{2j}y_2 + \cdots + C_{nj}y_n \Rightarrow \Phi = \Psi C \text{ for } C = (C_{ij}).$$

Proposition: Suppose we are given a fundamental matrix Φ for $x' = Ax$ then we can calculate e^{At} as follows,

$$e^{At} = \Phi(t)\Phi^{-1}(0) \qquad \text{Eq}^{\underline{n}}\ (6)$$

Proof: we know e^{At} is also a fundamental matrix so $\exists$ C such that $e^{At} = \Phi(t)C$ by Lemma (1). Lets find C then consider $t=0$,

$$e^{0t} = I = \Phi(0)C \Rightarrow C = \Phi^{-1}(0) \therefore \underline{e^{At} = \Phi(t)\Phi^{-1}(0)}.$$

Remark: If we know $\Phi(t)$ then everything else is kinda-silly. I mean that's the whole problem, finding $\Phi(t)$ is finding n-LI sol$^{\underline{n}}$'s to $x' = Ax$. Now e^{At} is a fundamental matrix for $x' = Ax$, but the problem now is how to calculate e^{At} for abitrairy A. We actually don't need to do that directly, as we saw in the case $\exists$ n-LI eigenvectors we could multiply e^{At} by U to find

$$e^{At}[u_1|u_2|\cdots|u_n] = [e^{\lambda_1 t}u_1|e^{\lambda_2 t}u_2|\cdots|e^{\lambda_n t}u_n] = \Phi$$

We never had to calculate e^{At} yet by finding eigenvectors we found Φ from e^{At} by multiplying U. What should we do if $\nexists$ n eigenvectors?

Def^{n}/ Let A be an $n \times n$ constant matrix and let λ be an eigenvalue of A. A nontrivial vector u satisfying

$$(A - \lambda I)^k u = 0$$

for some positive integer k is called a <u>generalized</u> <u>eigenvector</u> of A with eigenvalue λ.

<u>Claim</u>: there always exist n-LI generalized eigenvectors for A. these fill the gap when we don't have enough eigenvectors. Of course eigenvectors are generalized eigenvectors with $k=1$. When $k>1$ things are more interesting.

<u>Proposition</u>: Let u be a generalized eigenvector for A then $e^{At}u$ is a sol^{n} to $x' = Ax$

<u>Observation</u>: We know $\exists \lambda$ with $(A-\lambda I)^k u = 0$ so consider,

$$\begin{aligned} e^{At}u &= e^{(A-\lambda I)t} e^{\lambda I t} u \\ &= e^{\lambda t} e^{(A-\lambda I)t} u \\ &= e^{\lambda t} \left\{ u + t(A-\lambda I)u + \tfrac{1}{2} t^2 (A-\lambda I)^2 u + \cdots + \frac{1}{(k-1)!} t^{k-1} (A-\lambda I)^{k-1} u \right\} \end{aligned}$$

- This isn't necessary for the proof, but it shows explicitly how to calculate $e^{At}u$ for a generalized eigenvector. Also when $k=1$ we recover $e^{At}u = e^{\lambda t}u$ as we found before for plain old eigenvectors.

<u>Proof</u>:
$$\begin{aligned} \frac{d}{dt}\left(e^{At}u\right) &= \frac{d}{dt}\left(e^{At}\right)u \\ &= Ae^{At}u \quad \therefore \ x = e^{At}u \text{ solves } x' = Ax. \end{aligned}$$

Lets summarize the strategy for finding the fundamental sol$^{\underline{n}}$ set for $x' = Ax$,

1.) Compute the characteristic polynomial for A; $p(\lambda)$ which will have $k \leq n$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with multiplicities $m_1, m_2, \ldots, m_k$ where $m_1 + m_2 + \cdots + m_k = n$.

2.) Find m_i generalized eigenvectors for λ_i for each $i = 1, 2, \ldots, k$. This means first you should find all the eigenvectors you can then look for generalized eigenvectors until you get m_i total vectors.

3.) Compute the n-LI sol$^{\underline{n}}$'s to $x' = Ax$ each of the form $e^{At}u$ which again reduces to

$$e^{At}u = e^{\lambda t}\{u + t(A - \lambda I)u + \tfrac{1}{2}t^2(A - \lambda I)u + \cdots\}$$

and the $+\cdots$ indicates finitely many terms as we have discussed. In practice we rarely get past $k = 2$ or 3 and often $k = 1$ (ordinary eigenvectors) will give us all the sol$^{\underline{n}}$'s we need.

E3 We know that $y'' - 2y' + y = 0$ has $\lambda^2 - 2\lambda + 1 = (\lambda-1)(\lambda-1) = 0 \therefore \lambda = 1$ twice $\therefore y = c_1 e^t + c_2 te^t$.
We ought to be able to calculate the te^t as a result of a generalized eigenvector for the corresponding normal form DEq^{n}. Let us change to x_1 & x_2

$$x_1 = y \quad \text{and} \quad x_2 = y'$$

Thus we have $x_1' = y' = x_2$ and $y'' = x_2' = 2y' - y = 2x_2 - x_1$
That is in matrix form we have,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = -\lambda(2-\lambda) + 1 = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2 = 0$$

Its nice that the auxillary and characteristic polynomials are the same, I prove it in general in Ex # 41 of §9.5.

<u>Find $u_1 = (u,v)^T$ with $Au_1 = u_1$</u>

$$\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow \begin{matrix} v = u \\ -u + 2v = v \end{matrix} \Rightarrow u = v \therefore \boxed{\underbrace{u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{eigenvector}}}$$

<u>Find $u_2 = (u,v)^T$ with $(A + I)^2 u_2 = 0$</u>

$$A - I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow (A-I)^2 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So most any vector will do, lets pick $\boxed{u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \text{ generalized eigenvector}}$

Now the sol^{n}'s are $e^{At}u_1$ and $e^{At}u_2$,

$$\vec{x}_1 = e^{At}u_1 = e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{x}_2 = e^t\left\{u_2 + t\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = e^t\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right\}$$

Thus the general sol^{n} will be

$$x = c_1 e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^t\begin{bmatrix} 1-t \\ -t \end{bmatrix} = \boxed{\begin{bmatrix} c_1 e^t + c_2 e^t(1-t) \\ c_1 e^t - c_2 te^t \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}$$

$\boxed{E3}$ Continuing, lets return to $Y = X_1$ and $Y' = X_2$ we found

$$X_1 = c_1 e^t + c_2 e^t (1-t) = Y$$
$$X_2 = c_1 e^t - c_2 t e^t = Y'$$

At least $Y' = Y$, its consistent. Moreover observe that

$$Y = c_1 e^t + c_2 e^t (1-t) = \underline{(c_1 + c_2) e^t - c_2 t e^t = Y}$$

But this is just what we claimed to begin with (with $c_1 \rightarrow c_1 + c_2$ and $c_2 \rightarrow -c_2$ but they're arbitrairy so don't be distracted by them it's the same!)

Therefore, we have derived the "t" in the double root case again. This time our intial jump was supposing that e^{At} was a sol$^{\underline{n}}$, I like this derivation the best. The one we found via Laplace transforms assumed more I think.

- So, I just made a simple problem quite complicated, I hope you appreciate it, I have my reasons...

$\boxed{E4}$ Find the general sol^{n} to $x' = Ax$ where $A = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$.

$$\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & -1 \\ 4 & -3-\lambda \end{bmatrix}$$
$$= (1-\lambda)(-3-\lambda) + 4$$
$$= \lambda^2 + 2\lambda + 1$$
$$= (\lambda+1)^2 = 0 \Rightarrow \underline{\lambda = -1 \text{ twice}}$$

Let $u_1 = [u\ v]^T$ find u_1 such that $(A+I)u_1 = 0$

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{array}{c} 2u - v = 0 \\ \therefore\ v = 2u \end{array} \rightarrow \underline{u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ will work.}}$$

So we found <u>one</u> eigenvector, but we need <u>two</u> sol^{n}'s. So we need to find a generalized eigenvector u_2 that satifies $(A+I)^2 u_2 = 0$.

$$(A+I)^2 = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} = 0$$

as we could have suspected w/o calculating just on the basis of Cayley Hamilton Th^{m} $P(A) = 0$.

Basically any vector will do, <u>except</u> nonzero multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
So I'll pick $u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ because I can. (more on this next pages)

<u>Now find the sol^{n}'s</u>

from u_1 we find $\vec{x}_1 = e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

from u_2 calculate
$$\vec{x}_2 = e^{-t}\{1 + t(A+I)\}u_2$$
$$= e^{-t}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\right\}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= e^{-t}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$$
$$= \begin{bmatrix} e^{-t} + 2te^{-t} \\ 4te^{-t} \end{bmatrix}$$

Thus the general sol^{n} is $\boxed{x = c_1 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} e^{-t} + 2te^{-t} \\ 4te^{-t} \end{bmatrix}}$

$\boxed{E4}$ So this was # 35 of §9.5. If you look the answer is

$$\underbrace{\vec{X}_1 = e^{-t}\begin{bmatrix}1\\2\end{bmatrix}}_{\text{same as ours}} \quad \& \quad \underbrace{\vec{X}_2(t) = te^{-t}\begin{bmatrix}1\\2\end{bmatrix} + e^{-t}\begin{bmatrix}1\\1\end{bmatrix}}_{\text{looks quite different, what gives?}}$$

I checked our sol$^{\underline{n}}$ and it worked, so the text's sol$^{\underline{n}}$ is just another way of writing it. It all goes back to the weird way I chose u_2. We can check that our $\{\vec{X}_1, \vec{X}_2\}$ forms a fundamental sol$^{\underline{n}}$ set,

$$\vec{X}_1 = e^{-tA}u_1 = e^{-t}\begin{bmatrix}1\\2\end{bmatrix}$$
$$\vec{X}_2 = e^{-tA}u_2 = e^{-t}\begin{bmatrix}1+2t\\4t\end{bmatrix}$$

$$\Xi = \begin{bmatrix} e^{-t} & e^{-t}(1+2t) \\ 2e^{-t} & e^{-t}(4t) \end{bmatrix}$$

$$W[\vec{x}_1, \vec{x}_2] = \det(\Xi) = e^{-2t}\big(4t - 2(1+2t)\big) = -2e^{-2t} \neq 0$$

So we know (since I checked) that $\vec{X}_2$ and $\vec{X}_1$ are sol$^{\underline{n}}$'s and they are LI by the Wronskian test. So where did the book get it's answer? Well it all goes back to the strategies given on pg. 543 for finding generalized eigenvectors. The book suggests we should choose u_2 so that it satisfies

$$(A+I)u_2 = u_1$$

Then u_2 is automatically a sol$^{\underline{n}}$ to $(A+I)^2 u_2 = 0$ since $(A+I)u_1 = 0$, see below.

$$(A+I)(A+I)u_2 = (A+I)u_1 = 0.$$

Lets find the books answer.

$\boxed{E4}$ Continued, we're attempting to uncover the book's logic for the answer to #35 §9.5, We know $u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ lets find $u_2 = \begin{bmatrix} u \\ v \end{bmatrix}$ with

$$(A+I)u_2 = u_1$$

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \left.\begin{array}{c} 2u - v = 1 \\ \Rightarrow v = 2u-1 \\ \text{let } u = 1 \end{array}\right\} \Rightarrow u_2 = \begin{bmatrix} 1 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then we can calculate the sol^n from u_2,

$$\begin{aligned}
\vec{X}_2 &= e^{tA} u_2 \\
&= e^{-t}\left\{ Iu_2 + t\underbrace{(A+I)u_2}_{} + \tfrac{1}{2}t^2(A+I)^2 u_2 + \cdots \right\} \\
&= e^{-t}\left\{ Iu_2 + t u_1 \right\} \\
&= e^{-t}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \\
&= \underbrace{te^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{the book's sol}^n.}
\end{aligned}$$

The advantage of doing it the way the book suggests is that if we demand that $(A+I)u_2 = u_1$ then u_2 is automatically LI from u_1 (we choose $u_2 \neq 0$) and when we calculate the sol^n from the matrix exponential there is less new stuff to calculate since we already know $(A+I)u_2 = u_1$.

CHAINS OF GENERALIZED EIGENVECTORS

We claimed that there always exist n generalized eigenvectors for <u>any</u> $n \times n$ matrix A. The proof of this is not trivial, one method involves the "Jordon Cannonical Form", take ma 520 if you want to see the details. So accept my claim they exist, we're still left with the problem of how to find them. Let me outline the idea of the <u>CHAIN</u>. Suppose that $P(A) = (A-\lambda)^m \dots$ then we know $\exists$ m-generalized eigenvectors. We proceed as follows

1.) find u_1 with $(A-\lambda I)u_1 = 0$

2.) find u_2 with $(A-\lambda I)u_2 = u_1$
notice that $(A-\lambda I)^2 u_2 = (A-\lambda I)u_1 = 0$.
Also note that $u_2 \neq c u_1$ since otherwise $u_1 = 0$ which is impossible.

3.) find u_3 with $(A-\lambda I)u_3 = u_2$ then automatically $(A-\lambda I)^3 u_3 = 0$ and we see that u_3 must be LI from u_2 and u_1 because otherwise we could argue u_1 and u_2 were trivial.

And so on... this method has the advantage that because of the extra conditions $(A-\lambda I)u_p = u_{p-1}$ we are gaurranteed the LI of $\{u_1, u_2, \dots, u_m\}$ and $e^{tA}u_m$ calculates nicely.

$$e^{tA}u_3 = e^{\lambda t}\{u_3 + t(A-\lambda I)u_3 + \tfrac{1}{2}t^2(A-\lambda I)^2 u_3\}$$
$$= e^{\lambda t}\{u_3 + t u_2 + \tfrac{1}{2}t^2 u_1\}$$

for example

$\boxed{E5}$ Find sol^{n} to $x' = Ax$ for $A = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ (#37 of §9.5)

$$\det[A - \lambda I] = \det\begin{bmatrix} 2-\lambda & 1 & 6 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)[(2-\lambda)^2] = 0 \quad \therefore \ \underline{\lambda = 2 \text{ three times}}$$

Find Eigenvectors with $\lambda = 2$

$$(A-2I)u_1 = \begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} v + 6w = 0 \\ 5w = 0 \end{matrix}$$

So notice $v = -6w$ and $w = 0$ thus $v = w = 0$, but u is left free so we choose $u = 1$ to keep things pretty

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Find Generalized Eigenvector u_2 with $(A-2I)u_2 = u_1$

$$(A-2I)u_2 = \begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} v + 6w = 1 \\ 5w = 0 \end{matrix} \Rightarrow \begin{matrix} v = 1 \\ w = 0 \end{matrix}$$

Again u is left free so choose $u = 0$ yielding

$$u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Find Generalized Eigenvector u_3 with $(A-2I)u_3 = u_2$

$$(A-2I)u_3 = \begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} v + 6w = 0 \\ 5w = 1 \end{matrix} \Rightarrow \begin{matrix} v = -6w = -6/5 \\ w = 1/5 \end{matrix}$$

Yet again u is left free so choose $u = 0$ yielding

$$u_3 = \begin{bmatrix} 0 \\ -6/5 \\ 1/5 \end{bmatrix}$$

We have a chain of generalized eigenvectors,

$$(A-2I)^3 u_3 = (A-2I)^2 u_2 = (A-2I)u_1 = 0$$

Remark: We never had to actually square or cube the matrix $A-2I$. That is another calculational advantage of the chain.

E5 Now find the sol^n's generated by the generalized eigenvectors we found.

$$\vec{X}_1 = e^{tA}u_1 = e^{-2t}(I + t\cancel{(A-2I)}^{0} + \cdots)u_1 = e^{-2t}u_1$$

$$\vec{X}_2 = e^{tA}u_2 = e^{-2t}(I + t(A-2I) + \tfrac{1}{2}t^2\cancel{(A-2I)^2}^{0} + \cdots)u_2$$
$$= e^{-2t}(u_2 + t(A-2I)u_2$$
$$= e^{-2t}(u_2 + tu_1)$$

$$\vec{X}_3 = e^{tA}u_3 = e^{-2t}(I + t(A-2I) + \tfrac{1}{2}t^2(A-2I)^2 + \tfrac{1}{3!}t^3\cancel{(A-2I)^3}^{0} + \cdots)u_3$$
$$= e^{-2t}(u_3 + tu_2 + \tfrac{1}{2}t^2u_1)$$

Then the general sol^n has the form, putting things together,

$$X = c_1\vec{X}_1 + c_2\vec{X}_2 + c_3\vec{X}_3$$
$$= c_1e^{-2t}\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2e^{-2t}\begin{bmatrix}t\\1\\0\end{bmatrix} + c_3e^{-2t}\begin{bmatrix}\tfrac{1}{2}t^2\\-6/5+t\\1/5\end{bmatrix}$$
$$= \begin{bmatrix}c_1e^{-2t} + c_2te^{-2t} + c_3e^{-2t}\tfrac{1}{2}t^2\\ c_2e^{-2t} + c_3(t-6/5)e^{-2t}\\ 1/5\,c_3e^{-2t}\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

We say $x = [x_1\ x_2\ x_3]^T$ usually so I put a vector on the sol^n $\vec{X}_1$ to avoid confusion. We've solved

$$\begin{aligned} x_1' &= 2x_1 + x_2 + 6x_3 \\ x_2' &= 2x_2 + 5x_3 \\ x_3' &= 2x_3 \end{aligned} \quad \Rightarrow \quad \boxed{\begin{aligned} x_1 &= c_1e^{-2t} + c_2te^{-2t} + \tfrac{1}{2}c_3t^2e^{-2t} \\ x_2 &= c_2e^{-2t} + c_3(t-6/5)e^{-2t} \\ x_3 &= \tfrac{1}{5}c_3e^{-2t} \end{aligned}}$$

Just wanted to write the end to remind us of what we have accomplished through all these weird calculations. I'm pretty sure I'd never have found this sol^n w/o the generalized eigenvector technique.

$\boxed{E6}$ Solve $x' = Ax$ for $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix}$ (#39 of §9.5)

$$\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ -2 & -2 & -1-\lambda \end{bmatrix}$$

$$= (2-\lambda)\left[(2-\lambda)(-1-\lambda)+2\right] - \left[-1-\lambda+2\right] + \left[-2+2(2-\lambda)\right]$$

$$= (2-\lambda)\left[-\lambda+\lambda^2\right] + \lambda - 1 + 2 - 2\lambda$$

$$= (2-\lambda)\lambda(\lambda-1) - (\lambda-1)$$

$$= (\lambda-1)\left[\lambda(2-\lambda)-1\right]$$

$$= (\lambda-1)(-\lambda^2+2\lambda-1)$$

$$= -(\lambda-1)^3 = 0 \qquad \therefore \ \underline{\lambda = 1 \text{ three times}}$$

<u>Find u_1 with $(A-I)u_1 = 0$</u>

$$(A-I)u_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$u+v+w=0$
really just one eq$^{\underline{n}}$ for three unknown $\Rightarrow$ two free parameters

lets use u & v as the free ones then $w = -u-v$. Then

$$u_1 = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u \\ v \\ -u-v \end{bmatrix} = u\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + v\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

We find two eigenvectors in this case, we'll call them

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

<u>Find u_3 with $(A-I)u_3 = u_1$</u>

$$(A-I)u_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$u+v+w = 1$
$u+v+w = 0$

$\therefore \ 0 = 1$ grrr... ☹

So we can't just use the chain in this case, we need to think more...

Need to use another eigenvector instead of u_1 or u_2, they'll both lead to inconsistent $(A-I)u_3 = u_1$ or u_2
we need something with same pattern as $A-I$, itself. By inspection try using $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$
notice that this is simply u_1+u_2 so it's an eigenvector.

E6 Continued, solve

$$(A-I)u_3 = u_1 + u_2$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \longrightarrow \left.\begin{array}{r} u+v+w = 1 \\ u+v+w = 1 \\ -2u-2v-2w = -2 \end{array}\right\} \text{consistent} ☺$$

Then $w = 1-u-v$ which leads us to

$$u_3 = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u \\ v \\ 1-u-v \end{bmatrix} = u\underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{u_1} + v\underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{u_2} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So we should redefine $u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as the u_1 & u_2 pieces are eliminated by the choice $u = v = 0$. We should have expected to only find one new generalized eigenvector here since we already have two LI eigenvector. (Linear Algebra explains that we can have at most 3 LI vectors with 3 components)

Remark: when there is more than just one eigenvector we find that the details of forming the chain are not as simple. The guiding principles are the same, but I don't claim to have given the general algorithm here. (ask me if your interested)

notice
$$\begin{aligned} (A-I)^2 u_3 &= (A-I)(u_1+u_2) \\ &= (A-I)u_1 + (A-I)u_2 \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

E6 Finished,

$$\vec{X}_1 = e^{At} u_1 = e^t u_1$$
$$\vec{X}_2 = e^{At} u_2 = e^t u_2$$
} eigenvector sol^{n}'s

$$\vec{X}_3 = e^{At} u_3$$
$$= e^t \{ I + t(A-I) + \tfrac{1}{2}t^2(A-I)^2 + \cdots \} u_3$$
$$= e^t \{ u_3 + t(A-I)u_3 \}, \text{ since } (A-I)^2 u_3 = 0.$$
$$= e^t \{ u_3 + t(u_1+u_2) \}, \text{ as } (A-I)u_3 = u_1 + u_2.$$

Then the general sol^{n} is

$$\boxed{\vec{X} = c_1 e^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 e^t \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}}$$

- the instructions given by the text for #39 §9.5 are motivated by reasoning similar to what I have given.

//

Remark: It's nice we can solve any system of the form $X' = Ax$, it's just a matter of work. But think about it another way, we could turn this around and use known differential eq^{n}'s to create matrices with special properties. Think about this point when you take linear algebra. This trick will allow you to create examples of matrices with generalized eigenvectors...

(this remark is not for now, I put it here mostly to remind myself.)