UNDETERMINED COEFFICIENTS

We wish to solve $y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_n y = g(x)$. This is a nonhomogeneous DEq^{n} and we know that the general sol^{n} will have the form

$$y = y_p + c_1 y_1 + \cdots + c_n y_n$$

where $y_1, y_2, \ldots, y_n$ are sol^{n}'s to the auxillary eq^{n} and <u>y_p is the particular sol^{n}</u>. If $g(x)$ is comprised of products of the functions that make up the homogeneous sol^{n} (that is polynomials, exponentials or sines & cosines) then y_p will have a form <u>similar</u> to $g(x)$.

METHOD OF UNDETERMINED COEFFICIENTS

- To find y_p the particular sol^{n} to a constant coeff. linear ODE $L[y] = Cx^m e^{rx}$ we guess

$$y_p = x^s[A_m x^m + \cdots + A_1 x + A_0]e^{rx}$$

with $s = 0$ if r is not a root of the auxillary eq^{n} and otherwise $s =$ multiplicity of the root in the aux. eq^{n}.

- If $L[y] = Cx^m e^{\alpha x}\cos\beta x$ or $Cx^m e^{\alpha x}\sin\beta x$ then use

$$y_p = x^s[A_m x^m + \cdots + A_1 x + A_0]e^{\alpha x}\cos\beta x + x^s[B_m x^m + \cdots + B_0]e^{\alpha x}\sin\beta x$$

with $s = 0$ if $\alpha + i\beta$ is not a root of the auxillary eq^{n} and $s =$ multiplicity of $\alpha + i\beta$ otherwise.

<u>Terminology</u>: $A_m, \ldots, A_1, A_0$ and $B_m, \ldots, B_0$ are the so-called undetermined coefficients. Upon completion of the method we through substituting y_p back into $L[y_p](x) = g(x)$ and then we use comparing coefficients to find eq^{n}'s to solve for the undet. coeff.

<u>Remark</u>: When $s = 0$ I refer to these as the "naive" guess for y_p. When $s \neq 0$ I say it is from "overlap". Basically you can construct these from demanding pairwise LI of all the functions in the homogeneous (aka aux) sol^{n} and y_p.

<u>FACT</u>: Superposition will allow us to treat $g(x) = g_1(x) + g_2(x)$ by attacking $g_1(x)$ then $g_2(x)$. More later on this idea.

$\boxed{E1}$ $y'' + y = t\cos t$

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm 1 \Rightarrow y_h = c_1\cos t + c_2\sin t$

Thus there is overlap and multiplicity of i is 1.

$$y_p = t[At + B]\cos t + t[Ct + D]\sin t$$

$$(*) \quad y_p = (At^2 + Bt)\cos t + (Ct^2 + Dt)\sin t$$

$$y_p' = (2At + B)\cos t - (At^2 + Bt)\sin t + (2Ct + D)\sin t + (Ct^2 + Dt)\cos t$$

$$y_p' = (2At + B + Ct^2 + Dt)\cos t + (2Ct + D - At^2 - Bt)\sin t$$

$$y_p'' = (2A + 2Ct + D)\cos t - (2At + B + Ct^2 + Dt)\sin t$$
$$+ (2C - 2At - B)\sin t + (2Ct + D - At^2 - Bt)\cos t$$

$$(**) \quad y_p'' = (2A + 2Ct + D + 2Ct + D - At^2 - Bt)\cos t + 2$$
$$+ (-2At - B - Ct^2 - Dt + 2C - 2At - B)\sin t$$

$$t\cos t = y_p'' + y_p = (\cancel{At^2} + \cancel{Bt} + 2A + 2Ct + D + 2Ct + D - \cancel{At^2} - \cancel{Bt})\cos t$$
$$+ (\cancel{Ct^2} + \cancel{Dt} - 2At - B - \cancel{Ct^2} - \cancel{Dt} + 2C - 2At - B)\sin t$$

To get the last eq$^{\underline{n}}$ -I just substituted $(*)$ & $(**)$ into the DEq$^{\underline{n}}$.
Now compare coefficients to find,

$\underline{\cos t}|$: $2A + 4Ct + 2D = t$

$\underline{\sin t}|$: $-4At - 2B + 2C = 0$

$\underline{t\cos t}|$: $4C = 1$

$\underline{\cos t}|$: $2A + 2D = 0$

$\underline{t\sin t}|$: $-4A = 0$

$\underline{\sin t}|$: $-2B + 2C = 0$

Hmm... many times I've jumped straight to the 4 eq$^{\underline{n}}$'s but breaking it into stages might organize better, whatever works for you. Anyway those eq$^{\underline{n}}$'s aren't hard to solve

$\hookrightarrow$ $C = 1/4$ and $A = 0$ (almost obvious)
$D = 0$ and $B = 1/4$ (follow from line above)

Thus

$$y_p = \frac{1}{4}t\cos t + \frac{1}{4}t^2\sin t$$

Yielding general sol$^{\underline{n}}$,

$$\boxed{y = c_1\cos t + c_2\sin t + \frac{1}{4}t\cos t + \frac{1}{4}t^2\sin t}$$

Summary: Find y_h, Guess y_p, determine coefficients then assemble general sol$^{\underline{n}}$ as $y = y_h + y_p$.

E2 $y'' = x^2$ with $y(0) = 0$ and $y'(0) = 1$

$\lambda^2 = 0 \Rightarrow y_h = c_1 + c_2 x$ so it overlaps and $s = 2$.

$$y_p = x^2[Ax^2 + Bx + C] = Ax^4 + Bx^3 + Cx^2$$
$$y_p' = 4Ax^3 + 3Bx^2 + 2Cx$$
$$y_p'' = 12Ax^2 + 6Bx + 2C$$

$$y_p'' = 12Ax^2 + 6Bx + 2C = x^2 \qquad (\text{Subst. } y_p'' \text{ into DEq}^{n})$$

Compare coefficients to obtain,

$x^2]$ $12A = 1$ $\quad A = 1/12$

$x]$ $6B = 0 \Rightarrow B = 0 \Rightarrow y_p = \frac{1}{12}x^4$

$1]$ $2C = 0$ $\quad C = 0$

Thus the general sol^{n} is $y = c_1 + c_2 x + \frac{1}{12}x^4$. But, we don't want the general sol^{n} we wish to find the specific sol^{n} that satisfies the intial data,

$$y(0) = c_1 = 0$$
$$y'(0) = c_2 + \tfrac{1}{3}(0) = 1 \Rightarrow \boxed{y = x + \tfrac{1}{12}x^4}$$

Remark: we could have found the general sol^{n} by integrating twice directly.

$$\int \frac{d}{dx}\frac{d}{dx}(y)\,dx = \int x^2 dx \Rightarrow \frac{dy}{dx} = \frac{1}{3}x^3 + c_2$$

$$\int \frac{dy}{dx}dx = \int\left(\tfrac{1}{3}x^3 + c_2\right)dx \Rightarrow y = \frac{1}{12}x^4 + c_2 x + c_1$$

This example is special because there is no y or y' think about why we cannot just straight away integrate *most* DEq^{n}'s (seeing this attempted on tests makes me grimace, unless of course it works)

Remark: Also notice we apply intial conditions to the general sol^{n}, *not* just the auxillary sol^{n} alone.

Annihilator Method

In short, this gives a proof of why our guesses for Y_p work. Our goal is to transform a nonhomogeneous ODE $L[Y](x) = g(x)$ into a corresponding homogeneous eq$^{\underline{n}}$ $AL[Y](x) = 0$. We will then find Y_p and Y_h for the original eq$^{\underline{n}}$ residing in the general sol$^{\underline{n}}$ for $AL[Y](x) = 0$.

$\boxed{E1}$ $Y'' + 2Y' + Y = e^{-x}$ here $L = D^2 + 2D + 1 = (D+1)^2$
We note $(D+1)e^{-x} = -e^{-x} + e^{-x} = 0$ so this suggests we choose $A = D+1$ so that $AL[Y] = A[e^{-x}] = 0$, notice,

$$AL = (D+1)(D+1)^2 = (D+1)^3$$

That is $AL[Y] = 0$ has auxillary eq$^{\underline{n}}$ $(\lambda+1)^3 = 0$ so $\lambda = -1$ with multiplicity 3 hence

$$\boxed{Y = \underbrace{C_1 e^{-x} + C_2 x e^{-x}}_{\text{auxillary sol}^{\underline{n}} \text{ to } L[Y] = e^{-x}} + \underbrace{C_3 x^2 e^{-x}}_{\substack{\text{correct } Y_p \text{ guess, complete} \\ \text{with the } x^2 \text{ to adjust for overlap.}}}}$$

- to complete the problem we'd subst. $C_3 x^2 e^{-x}$ back in to $L[Y] = e^{-x}$ and determine C_3.

$\boxed{E2}$ $Y'' = xe^x + \sin x$ here $L = D^2$ we need to find the annihilator A of $xe^x + \sin x$

$$(D^2+1)\sin x = -\sin x + \sin x = 0$$
$$(D-1)^2 xe^x = 0$$

(see page ____)

Hence $A = (D^2+1)(D-1)^2$ then $AL = D^2(D^2+1)(D-1)^2$ which means $AL[Y] = 0 \Rightarrow$

$$Y = \underbrace{C_1 + C_2 x}_{Y_h} + \underbrace{C_3 \cos x + C_4 \sin x + C_5 e^x + C_6 x e^x}_{\text{correct guess for } Y_p}$$

Remark: $\boxed{E2}$ illustrates the superposition principle. The total guess for $Y_p = Y_{p_1} + Y_{p_2}$ where $Y_{p_1} = C_3 \cos x + C_4 \sin x$ stems from $g_1(x) = \sin(x)$ and $Y_{p_2} = C_5 e^x + C_6 x e^x$ stems from $g_2(x) = xe^x$. In view of the superposition principle we can treat any nonhomogeneous term which is a sum and/or product of polynomials, exponentials or sines or cosines with the METHOD OF UNDET. COEFFICIENTS. What about $g(x) = \tan(x)$? We need a new approach...

Variation of Parameters

Consider $y^{(n)}(x) + P_1(x)\,y^{(n-1)}(x) + \cdots + P_n(x)\,y(x) = g(x)$ $\quad$ Eq$^{\underline{n}}$ (1)
Suppose that it has fundamental sol$^{\underline{n}}$ set to auxillary eq$^{\underline{n}}$ $\{y_1, y_2, \ldots, y_n\}$ then we know the general sol$^{\underline{n}}$ has the form $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n + y_p$. Our task is to find the particular sol$^{\underline{n}}$ (here $g(x)$ need not be one of the functions that worked for undet. coeff.) We guess that if we bump-up the constant undetermined coeff. to undetermined functions then we can treat more cases, That is suppose,

$$\boxed{y_p = V_1(x)\,y_1(x) + V_2(x)\,y_2(x) + \cdots + V_n(x)\,y_n(x)}$$

where $V_1, V_2, \ldots, V_n$ are unknown functions we must somehow find. Your text derives that the functions can be calculated by

$$V_k(x) = \int \frac{g(x)\,W_k(x)\,dx}{W[y_1, y_2, \ldots, y_n](x)}$$

Where W is the Wronskian and W_k is the related Wronskian defined by $W_k(x) = (-1)^{n-k}\,W[y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n](x)$.

Remark: In 6.4 # 2 I implemented these formulas to solve for y_p. I found in that calculation that $W_2(x) = -(x+1)e^x$. At first I found this disconcerting since $W_2(-1) = 0$, yet $\{1, xe^x\}$ are L.I. functions so why was the Wronskian $= 0$. (see Th$^{\underline{m}}$(3) on pg. 322). The answer is simple, the functions 1 and xe^x do not form a solution set for some DEq$^{\underline{n}}$. Whereas $\{1, e^x\}$ and $\{e^x, xe^x\}$ are sol$^{\underline{n}}$ sets to $y'' - y' = 0$ and $y'' - 2y' + y = 0$ respective. That is why we are guaranteed that $W_1(x)$ and $W_3(x)$ are never zero. In summary $W_k(x) = 0$ for some x is no problem because $W_k(x)$ is not necessarily formed from a fundamental sol$^{\underline{n}}$ set.

$\boxed{E1}$ Find general sol^n to the Cauchy - Euler eq^n:

$$x^3 y''' + x^2 y'' - 2x y' + 2y = x^3 \sin(x) \quad \text{for } x > 0$$

this example is a Cauchy - Euler eq^n, divide by x^3 to get

$$y''' + \frac{1}{x} y'' - \frac{2}{x^2} y' + \frac{2}{x^3} y = \sin(x)$$

The corresponding homogeneous eq^n $y''' + \frac{1}{x} y'' - \frac{2}{x^2} y' + \frac{2}{x^3} y = 0$ has fundamental sol^n set $\{x, x^{-1}, x^2\}$ (you can check this) thus we wish to find the functions v_1, v_2 and v_3 such that

$$y_p = v_1 x + v_2 \frac{1}{x} + v_3 x^2$$

is a sol^n. Calculate the Wronskian of $\{x, x^{-1}, x^2\}$,

$$W[x, x^{-1}, x^2](x) = \begin{vmatrix} x & x^{-1} & x^2 \\ 1 & -1/x^2 & 2x \\ 0 & 2/x^3 & 2 \end{vmatrix} = x\left(-2/x^2 - 4/x^2\right) - \cancel{\tfrac{1}{x}(2)} + \cancel{x^2(2/x^3)} = -6/x$$

Then we calculate $W_k(x)$ for $k = 1, 2, 3$

$$W_1(x) = (-1)^{3-1} W[x^{-1}, x^2](x) = \begin{vmatrix} x^{-1} & x^2 \\ -1/x^2 & 2x \end{vmatrix} = 2 - (-1) = 3$$

$$W_2(x) = (-1)^{3-2} W[x, x^2](x) = -\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = -2x^2 + x^2 = -x^2$$

$$W_3(x) = (-1)^{3-3} W[x, x^{-1}](x) = \begin{vmatrix} x & x^{-1} \\ 1 & -1/x^2 \end{vmatrix} = -1/x - 1/x = -2/x$$

From which we calculate,

$$v_1(x) = \int \frac{\sin(x) \cdot 3}{-6/x} dx = -\frac{1}{2} \int x \sin(x) dx = -\frac{1}{2} \sin(x) + \frac{1}{2} x \cos(x) + C_1$$

$$v_2(x) = \int \frac{\sin(x)(-x^2)}{-6/x} dx = \frac{1}{6} \int x^3 \sin(x)\, dx = (6x - x^3)\cos(x) + 3(x^2 - 2)\sin(x) + C_2$$

$$v_3(x) = \int \frac{\sin(x)(-2/x)}{-6/x} dx = \frac{1}{3} \int \sin(x) dx = -\frac{1}{3} \cos(x) + C_3$$

Combining these results yields,

$$\boxed{y_p = \cos(x) - x^{-1} \sin(x) + C_1 x + C_2 x^{-1} + C_3 x^2}$$

note $C_1 x + C_2 x^{-1} + C_3 x^2$ is the homogeneous sol^n if we neglect to add constants of integration it will not make y_p incorrect.

Remark: Cauchy - Euler Eq^n's can be solved by supposing the sol^n has form $y = x^k$ then we'll get a polynomial in k (just like const. coeff's λ) For example substituting into this problem aux. Deq^n yields,

$$\left(k(k-1)(k-2) + k(k-1) - 2k + 2\right) x^{k-3} = 0$$

$$(k-1)\left[k(k-2) + k - 2\right] = (k-1)(k^2 - k - 2) = \underbrace{(k-1)(k-2)(k+1)}_{\{x^1,\, x^2,\, x^{-1}\}} = 0$$

VARIATION OF PARAMETERS (JUST $n=2$)

We study $\boxed{aY''+bY'+cY = g(x)}$ Eq^{n}(1) which has fundamental sol^{n} set $\{y_1, y_2\}$. We suppose that there exists a sol^{n} of the form

$$\boxed{Y_p = V_1 y_1 + V_2 y_2} \qquad \text{Eq}^{n}(2)$$

Let us <u>derive</u> conditions on V_1 and V_2 to insure that Eq^{n}(2) does indeed provide a sol^{n} to Eq^{n}(1).

$$Y_p' = V_1' y_1 + V_1 y_1' + V_2' y_2 + V_2 y_2' \underset{\text{(using constraint)}}{=\!=} V_1 y_1' + V_2 y_2'$$

We impose the <u>constraint</u> $y_1 V_1' + y_2 V_2' = 0$ so that we will not get V_1'' and V_2'' in Y_p''. You might ask what right do we have to <u>add</u> this constraint? The answer is just that we are looking for a sol^{n} that works, so if this added constraint doesn't get in the way of that then we're good to go. You'll notice the text is not to verbose on this point. I think in general something is probably lost by adding this constraint, however for a large class of problems the method works. A more fundamental question to ask is why should Eq^{n}(2) be the only form for Y_p to take, maybe Y_p cannot be encapsulated by that ansatz either. My musing aside lets continue,

$$Y_p'' = V_1' y_1' + V_1 y_1'' + V_2' y_2' + V_2 y_2''$$

Thus,

$$\begin{aligned} g &= aY_p'' + bY_p' + cY_p \\ &= a(V_1'y_1' + V_1 y_1'' + V_2' y_2' + V_2 y_2'') + b(V_1 y_1' + V_2 y_2') + c(V_1 y_1 + V_2 y_2) \\ &= V_1 \underbrace{(ay_1'' + by_1' + cy_1)}_{=0} + V_2 \underbrace{(ay_2'' + by_2' + cy_2)}_{=0} + a(V_1' y_1' + V_2' y_2') \end{aligned}$$

(y_1 & y_2 solve the aux. eq^{n})

The calculations on this page reveal that $Y_p = V_1 y_1 + V_2 y_2$ will solve Eq^{n}(1) provided V_1 & V_2 satisfy

$$\boxed{\begin{aligned} 0 &= y_1 V_1' + y_2 V_1' \\ g/a &= y_1' V_1' + y_2' V_2' \end{aligned}} \qquad \text{Eq}^{n}(9)$$

where $\{y_1, y_2\}$ are the fundamental sol^{n} set.

We found that $y_p = v_1 y_1 + v_2 y_2$ satisfied eq$^{\underline{n}}$ (9) which can be rewritten in matrix form

$$\underbrace{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} v_1' \\ v_2' \end{bmatrix}}_{v'} = \underbrace{\begin{bmatrix} 0 \\ g/a \end{bmatrix}}_{b}$$

This is restated $Av' = b$ which has sol$^{\underline{n}}$ $v' = A^{-1}b$ and we can calculate the inverse easily here,

$$A^{-1} = \frac{1}{y_1 y_2' - y_2 y_1'} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} = \frac{1}{W[y_1, y_2]} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}$$

Identifying that the denominator is in fact the Wronskian of y_1 & y_2.

$$v' = \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \frac{1}{W[y_1, y_2]} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ g/a \end{bmatrix}$$

$$= \frac{1}{W[y_1, y_2]} \begin{bmatrix} -y_2 g/a \\ y_1 g/a \end{bmatrix}$$

Which gives us two eq$^{\underline{n}}$'s

$$\frac{dv_1}{dx} = \frac{-y_2 g}{a\,W[y_1, y_2]} \qquad \frac{dv_2}{dx} = \frac{y_1 g}{a\,W[y_1, y_2]}$$

Notice that $W_1(x) = -y_2 g$ whereas $W_2(x) = y_1 g$ so this really does match what we saw in §6.4. Integrating yields

$$\boxed{\begin{aligned} v_1 &= \int \frac{-y_2 g}{a(y_1 y_2' - y_2 y_1')}\, dx \\ v_2 &= \int \frac{y_1 g}{a(y_1 y_2' - y_2 y_1')}\, dx \end{aligned}} \qquad \text{Eq}^{\underline{n}}\ (10).$$

<u>Remark</u>: the derivation for the $n^{\underline{th}}$ order follows the same logic as here except that additional constraints are used and instead of computing A^{-1} the text uses Kramer's Rule to solve the analogue of eq$^{\underline{n}}$ (9) which is eq$^{\underline{n}}$ (7) of pg. 339.

E1 $y'' + y = \tan\theta$

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow y_1 = \cos\theta$ & $y_2 = \sin\theta$

Use eq^{n} (10) to calculate v_1 & v_2

$$v_1 = \int \frac{-y_2 g}{y_1 y_2' - y_2 y_1'} d\theta \quad , \text{ note } y_1 y_2' - y_2 y_1' = \cos^2\theta + \sin^2\theta = 1.$$

$$= \int -\sin\theta \tan\theta \, d\theta \quad , \text{ note } -\sin^2\theta = \cos^2\theta - 1$$

$$= \int \frac{\cos^2\theta - 1}{\cos\theta} d\theta$$

$$= \int (\cos\theta - \sec\theta) d\theta \quad , \text{ take } u = \sec\theta + \tan\theta \Rightarrow \frac{du}{u} = \sec\theta d\theta ,$$

$$= \boxed{\sin\theta - \ln|\sec\theta + \tan\theta| + C_1 = v_1}$$

$$v_2 = \int y_1 g \, d\theta$$

$$= \int \cos\theta \tan\theta \, d\theta$$

$$= \int \sin\theta \, d\theta$$

$$= \boxed{-\cos\theta + C_2 = v_2}$$

Hence we assemble the general sol^{n},

$$y = C_1\cos\theta + C_2\sin\theta + \cancel{\cos\theta\sin\theta} - \cos\theta\ln|\sec\theta + \tan\theta| - \cancel{\sin\theta\cos\theta}$$

That is, $\boxed{y = C_1\cos\theta + C_2\sin\theta - \cos\theta\ln|\sec\theta + \tan\theta|}$