GENERAL SOLUTION OF THE n^{th} ORDER CONSTANT COEFFICIENT ODEq^{n}

- PAGES 17 → 30 OF MY NOTES EXPLORE THIS QUESTION AND MORE, THE PURPOSE OF THESE NOTES IS TO EMPHASIZE THE LOGIC WITHIN THOSE NOTES, HERE I'll SIMPLY COLLECT THE MAIN IDEAS AND ASSEMBLE THEM TO DERIVE THE SOL^{n} TO THE n^{th} ORDER LINEAR CONST. COEFF. ODE.

GOAL: Given that $a_0, a_1, a_2, \dots, a_n$ are real constants find general sol^{n} of

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0 \qquad \text{Eq}^{n}\text{ (I)}$$

EXAMPLE: $y'' + 5y' + 6y = 0$ { Many of you did these in calc II }

$\lambda^2 + 5\lambda + 6 = 0$

$(\lambda+3)(\lambda+2) = 0 \quad \therefore \lambda_1 = -3, \ \lambda_2 = -2$

therefore (by things we'll derive today)

$$\boxed{y = c_1 e^{-3x} + c_2 e^{-2x}}$$

OBSERVATION (I): Eq^{n} (I) can be rewritten as an operator eq^{n}; $\boxed{L[y] = 0}$ where $L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_2 D^2 + a_1 D + a_0$ with $D \equiv d/dx$.

EXAMPLE: $y'' + 5y' + 6y = 0 \iff (D^2 + 5D + 6)y = 0$

$\iff L = D^2 + 5D + 6$ and $L[y] = 0$.

CLAIM (I): THE L FROM Eq^{n} (I) CAN BE FACTORED INTO n-factors $L = L_1 \circ L_2 \circ \cdots \circ L_n$ so $L[y] = L_1(L_2(\cdots(L_n(y))\cdots)) = 0$.

EXAMPLE: $L = D^2 + 5D + 6 = (D+3)(D+2) = L_1 L_2$

OBSERVATION ②: If $L_k[y] = 0$ for some particular k with $1 \le k \le n$ then y is also a sol$^{\underline{n}}$ to $L[y] = L_1 L_2 \dots L_n [y] = 0$.

Example: $y_1 = e^{-3x}$ solves $L_1[y] = (D+3)[y] = 0$ since $(D+3)[e^{-3x}] = -3e^{-3x} + 3e^{-3x} = 0$. Notice that $L[e^{-3x}] = (D+2)(D+3)e^{-3x} = (D+2)(0) = 0$.

Remark: OBSERVATION ② says that we can break-up the $n^{\underline{th}}$ order ODE into n-parts. If we can solve $L_k[y] = 0$ for $k = 1, 2, \dots, n$ then we'll get n-sol$^{\underline{n}}$'s to $Eq^{\underline{n}}$ ①.

CLAIM ②: ACTUALLY A FEW CLAIMS,

(i.) A linear combination of sol$^{\underline{n}}$'s to $L[y]=0$ is a sol$^{\underline{n}}$.

$$L[y_k] = 0 \quad k = 1, 2, 3, \dots, n \Rightarrow L[c_1 y_1 + c_2 y_2 + \dots + c_n y_n] = 0$$

where $c_1, c_2, \dots, c_n$ are constants.

(ii.) The general sol$^{\underline{n}}$ to $L[y] = 0$ has the form

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

where $y_1, y_2, \dots, y_n$ are n- "linearly independent" functions.

Example: $L = (D+3)(D+2) = L_1 L_2$ has $L_1[e^{-3x}] = 0$ and $L_2[e^{-2x}] = 0$ So $L[e^{-3x}] = 0$ and $L[e^{-2x}] = 0 \Rightarrow y = c_1 e^{-3x} + c_2 e^{-2x}$ solves $L[y]=0$. Moreover, since $y_1 = e^{-3x}$ and $y_2 = e^{-2x}$ are linearly independent we have the general sol$^{\underline{n}}$.

Remark: linear independence is an important concept which we elaborate on farther in future lectures. For now, simply take it to mean that the functions are distinct (they have graphs which have different shapes).

OBSERVATION 3: The L from Eq.^n I may be factored into the form
$$L = a_n(D-\lambda_n)(D-\lambda_{n-1})\cdots(D-\lambda_3)(D-\lambda_2)(D-\lambda_1)$$
where $\lambda_1, \lambda_2, \dots, \lambda_n$ are n-constants. These constants may be real or complex and possibly $\lambda_1 = \lambda_2$ etc.... If there are complex roots then they come in conjugate pairs. (if $\lambda_1 = 3+i$ then $\lambda_2 = 3-i$ for example)

CLAIM 3: for any $\lambda \in \mathbb{C}$, $(D-\lambda)[y] = 0$ has the sol^n. $y = e^{\lambda x}$

Proof:
$$\begin{aligned}(D-\lambda)[e^{\lambda x}] &= \tfrac{d}{dx}(e^{\lambda x}) - \lambda e^{\lambda x} \\ &= \lambda e^{\lambda x} - \lambda e^{\lambda x} \\ &= 0.//\end{aligned}$$

: I explain the details of when $\lambda \in \mathbb{C}$ on pg. 25. Short story, in $\mathbb{C}$ calculus works the same for functions $f: \mathbb{R} \to \mathbb{C}$.

CLAIM 4: $(D-\lambda)^2[y] = 0$ has the sol^n $y_2 = xe^{\lambda x}$ and $y_1 = e^{\lambda x}$

Proof:
$$\begin{aligned}(D-\lambda)^2[xe^{\lambda x}] &= (D-\lambda)\left[\tfrac{d}{dx}(xe^{\lambda x}) - \lambda xe^{\lambda x}\right] \\ &= (D-\lambda)[e^{\lambda x} + x\lambda e^{\lambda x} - \lambda xe^{\lambda x}] \\ &= (D-\lambda)[e^{\lambda x}] \\ &= 0.//\end{aligned}$$

CLAIM 5: $(D-\lambda)^m[y] = 0$ has m-sol^n's
$$y_m = x^{m-1}e^{\lambda x},\ y_{m-1} = x^{m-2}e^{\lambda x},\ \dots,\ y_2 = xe^{\lambda x},\ y_1 = e^{\lambda x}$$

Remark: these sol^n's $y_m, y_{m-1}, \dots, y_2, y_1$ are linearly independent. Sometimes they are complex sol^n's, we wish to find real sol^n's. This requires a little discussion

CLAIM: 6 If y is a complex-valued sol^{n} to $L[y]=0$ where $y = Re(y) + i\, Im(y)$ then both $y_1 = Re(y)$ and $y_2 = Im(y)$ are real-valued sol^{n}'s to $L[y]=0$. Hence, every complex sol^{n} has two (linearly independent) real sol^{n}'s.

Example: $y''+y=0 \iff (D^2+1)[y]=0$
$$\iff (D-i)(D+i)[y]=0$$
now $(D-i)[y]$ has sol^{n} $y=e^{ix}$. By Euler's Identity we have that $e^{ix} = \cos(x) + i\sin(x)$. Thus
$$y = \cos(x) + i\sin(x)$$
we can read off that $Re(y) = \cos(x)$ & $Im(y) = \sin(x)$.
These real sol^{n}'s give us the general sol^{n} $y = c_1\cos(x) + c_2\sin(x)$.

What about $(D+i)$ you ask? Well that factor gives the same sol^{n}'s since $(D+i)[y]=0 \Rightarrow y = e^{-ix}$.
then $y = e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x)$
and we can read-off $Re(y) = \cos(x)$ and $Im(y) = -\sin(x)$.
These are the "same" (upto linear independence) functions we already found from $(D-i)$.

Digression to answer common question.

PROPERTIES OF COMPLEX EXPONENTIAL

If $\lambda = \alpha + i\beta$ then we find
$$e^{\lambda x} = e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$$
then $e^{\lambda x} = e^{\alpha x}\cos\beta x + i e^{\alpha x}\sin\beta x$ and we can see
$$\boxed{Re(e^{\lambda x}) = e^{\alpha x}\cos\beta x \quad \Big| \quad Im(e^{\lambda x}) = e^{\alpha x}\sin\beta x}$$

(see the Intro. to Complex Notes for more details on Euler's Identity)

Remark: CLAIM 6 for $\lambda = \alpha + i\beta$ (here α, β are assumed to be real) we find two real sol$^{\underline{n}}$'s hidden inside $y = e^{\lambda x}$ namely

$$y_1 = \mathrm{Re}\, y = e^{\alpha x}\cos\beta x$$
$$y_2 = \mathrm{Im}\, y = e^{\alpha x}\sin\beta x$$

the general sol$^{\underline{n}}$ to $(D-\lambda)(D-\lambda^*)[y] = 0$ is $y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$.

(Here $\lambda^* = \alpha - i\beta$, the factor $(D-\lambda^*)$ has same sol$^{\underline{n}}$'s as $D-\lambda$ so it is sufficient to find the sol$^{\underline{n}}$'s to $D-\lambda$.)

again the digression

> CLAIM 7 If we have a complex sol$^{\underline{n}}$ $y = x^m e^{\lambda x}$ then there are two real sol$^{\underline{n}}$'s hidden inside y, namely Re y and Im y which we can easily calculate
>
> $$y_1 = \mathrm{Re}\, y = x^m e^{\alpha x}\cos\beta x$$
> $$y_2 = \mathrm{Im}\, y = x^m e^{\alpha x}\sin\beta x$$
>
> where again we have assumed $\lambda = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$.

Concluding Thoughts: we see Eq$^{\underline{n}}$ (I) can be rewritten in terms of a polynomial in $D = d/dx$ that is $L = P(D)$ where $P(D) = a_n D^n + \cdots + a_2 D^2 + a_1 D + a_0$. Then we factor P which of course reflects the zeroes of $P(D)$. For different types of factors we get either e^{ax}, $x^m e^{ax}$, $e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$ or $x^m e^{\alpha x}\cos\beta x$ or $x^m e^{\alpha x}\sin\beta x$. There will be n - of these functions which we can then use to form the general sol$^{\underline{n}}$.

Observation: We can factor $P(\lambda)$ instead of $P(D)$, get same roots!

$$\begin{aligned} P(e^{\lambda x}) &= a_n D^n e^{\lambda x} + \cdots + a_2 D^2 e^{\lambda x} + a_1 D e^{\lambda x} + a_0 e^{\lambda x} \\ &= a_n \lambda^n e^{\lambda x} + \cdots + a_2 \lambda^2 e^{\lambda x} + a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x} \\ &= (a_n \lambda^n + \cdots + a_2\lambda^2 + a_1\lambda + a_0)\, e^{\lambda x} \\ &= P(\lambda) e^{\lambda x} \quad \therefore \quad \boxed{P(e^{\lambda x}) = 0 \iff P(\lambda) = 0} \end{aligned}$$