Theory of Linear ODE's

These notes are based on chapters 4 & 6 of your text, rather than revisting the 2^{nd} order case alone, we treat it as a subcase of the n^{th} order ODE theory. I have notes on 2^{nd} order linear constant coefficient ODE's from ma 241, those are posted for review, although we will again discuss them here. Chapter 4 of the text does an excellent job on the 2^{nd} order case alone.

Def^n/ $Y^{(n)}(x) + P_1(x)\,Y^{(n-1)}(x) + \cdots + P_n(x)\,Y(x) = g(x)$ is a linear ordinary differential eq^n in standard form. $\quad (*)$

- We usually assume $P_i(x)$ and $g(x)$ are continous on some interval $I \subseteq \mathbb{R}$.

- when $P_1(x), P_2(x), \ldots, P_n(x)$ are all constants w.r.t. x this is said to be an n^{th} order <u>constant coefficient</u> ODE.

- when $g(x) = 0$ this is a <u>homogeneous</u> ODE, otherwise if $g(x) \neq 0$ for some $x \in I$ (not necessarily all of I, just a point or two will do it) we say the DEq^n $(*)$ is <u>nonhomogeneous</u>

- When $g(x)$ is nontrivial we define the <u>auxillary eq^n</u> to be,

$$\boxed{Y^{(n)}(x) + P_1(x)\,Y^{(n-1)}(x) + \cdots + P_n(x)\,Y(x) = 0}$$

Sol^n's to the auxillary eq^n play an important role in finding sol^n's to the DEq^n $(*)$. It is important to note that sol^n's to the auxillary eq^n are <u>not</u> sol^n's to $(*)$.

Th^{m}/ Suppose that $P_1, P_2, \dots, P_n, g$ are continous functions on (a, b) that contains x_0. Then the following IVP has a <u>unique</u> sol^{n} on (a, b).

$$\boxed{y^{(n)}(x) + P_1(x)\,y^{(n-1)} + \dots + P_n(x)\,y(x) = g(x)} \quad (*)$$

$$\left.\begin{aligned} y(x_0) &= \gamma_0 \\ y'(x_0) &= \gamma_1 \\ &\;\vdots \\ y^{(n-1)}(x_0) &= \gamma_{n-1} \end{aligned}\right\} \text{ intial conditions}$$

This is the "<u>existence</u> and <u>uniqueness</u>" theorem. It's a little involved to prove, and it doesn't really tell us how to find the sol^{n} either. However, it's nice to know that a problem can be solved and we can use this Th^{m} to prove other important facts.

<u>OPERATORS</u>

An operator is a generalization of our notion of ordinary functions. An operator takes functions and maps them to new functions

"usual" function
$$x \longrightarrow \boxed{\;f\;} \longrightarrow f(x) \qquad \begin{matrix} x \in \mathbb{R} \\ f(x) \in \mathbb{R} \end{matrix}$$

operator D
$$f \longrightarrow \boxed{\;D\;} \longrightarrow Df \qquad \begin{matrix} f : \mathbb{R} \to \mathbb{R} \\ Df : \mathbb{R} \to \mathbb{R} \end{matrix}$$

$$(Df)(x) \equiv \frac{d}{dx}(f(x))$$

Technically, D is also a function-valued function of functions, we call it an operator to draw attention to that fact. We can rewrite a DEq^{n} as an operator eq^{n}, define

$$L \equiv D^n + P_1 D^{n-1} + \dots + P_n \qquad \Big(\text{note } D^n = \underbrace{D \circ D \circ \dots \circ D}_{\text{composition}}\Big)$$

Then $(*)$ is simply $L[y] = g$. It is customary to use $[\ ]$ to emphasize the input is a function and L is an operator which means $L[y]$ is a new function.

The operator L is a <u>linear operator.</u> Meaning

Def^n of <u>Linear Operator</u>

$$\boxed{\begin{array}{l} 1.)\ L[Y_1 + Y_2 + \cdots + Y_m] = L[Y_1] + L[Y_2] + \cdots + L[Y_m] \\ 2.)\ L[cY] = c\,L[Y] \quad \text{for any constant } c. \end{array}}$$

<u>Remark</u>: calculus is full of linear operations, indefinite integration, differentiation, Laplace Transform, ...

Now suppose we have a <u>homogeneous</u> linear n^{th} ODE in standard form on some interval I. We can compactly write it as $L[Y] = 0$ or putting in the x-dependence $L[Y](x) = 0$. Further suppose we have 2 sol^{n}'s Y_1 and Y_2 notice,

$$\begin{aligned} L[C_1Y_1 + C_2Y_2] &= L[C_1Y_1] + L[C_2Y_2] && \text{used } 1. \\ &= C_1\,L[Y_1] + C_2\,L[Y_2] && \text{used } 2. \\ &= C_1\,(0) + C_2\,(0) && \text{using } Y_1 \text{ \& } Y_2 \text{ are sol}^n\text{s.} \\ &= 0. \end{aligned}$$

This proves $C_1Y_1 + C_2Y_2$ is also a sol^n. So then we might wonder how many "independent" sol^n's are there for an n^{th} order homog. linear ODE?

$$\boxed{\text{PROPOSITION: Every sol}^n \text{ to } L[Y](x) = 0 \text{ has the form } Y = C_1Y_1 + C_2Y_2 + \cdots + C_nY_n, \text{ where } L[Y_i](x) = 0 \quad i = 1, 2, \ldots, n}$$

- in other words an n^{th} order linear ODE has a general sol^n formed from n-<u>fundamental</u> sol^n's $Y_1, \ldots, Y_n$. Also $\{Y_1, Y_2, \ldots, Y_n\}$ is the fundamental sol^n set. Well, we need to require that $Y_1, Y_2, \ldots, Y_n$ are really not just the same function, I mean they should be genuinely independent like x and e^x or $\sin(x)$ and 3 as opposed to say x and $3x$, those are the same. (the functions $Y_1, \ldots, Y_n$ must be <u>linearly independent</u>, jumping ahead a bit.)

Proof of Prop.: Study $[D^n + P_1 D^{n-1} + \cdots + P_n][Y] = 0$. Suppose that we have an arbitrary sol$^{\underline{n}}$ $\phi(x)$ on (a,b). We seek to show $\phi(x) = c_1 Y_1 + c_2 Y_2 + \cdots + c_n Y_n$ for appropriate choice of $c_1, c_2, \ldots, c_n$ (constants). Let $x_0 \in (a,b)$ and consider that if we can satisfy the system of eq$^{\underline{n}}$'s,

$$\left.\begin{aligned} c_1 Y_1(x_0) + c_2 Y_2(x_0) + \cdots + c_n Y_n(x_0) &= \phi(x_0) \\ c_1 Y_1'(x_0) + c_2 Y_2'(x_0) + \cdots + c_n Y_n'(x_0) &= \phi'(x_0) \\ &\vdots \\ c_1 Y^{(n-1)}(x_0) + c_2 Y_2^{(n-1)}(x_0) + \cdots + c_n Y_n^{(n-1)}(x_0) &= \phi^{(n-1)}(x_0) \end{aligned}\right\} (**)$$

then by the existence & uniqueness Th$^{\underline{m}}$ we'll be able to conclude that $\phi(x) = c_1 Y_1 + \cdots + c_n Y_n$ as they both solve the same IVP. We can rewrite $(**)$ in matrix form

$$Ac = \underbrace{\begin{bmatrix} Y_1(x_0) & Y_2(x_0) & \cdots & Y_n(x_0) \\ \vdots & & & \vdots \\ Y_1^{(n-1)}(x_0) & Y_2^{(n-1)}(x_0) & \cdots & Y_n^{(n-1)}(x_0) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}_{c} = \underbrace{\begin{bmatrix} \phi(x_0) \\ \vdots \\ \phi^{(n-1)}(x_0) \end{bmatrix}}_{b} \equiv b$$

This is a system of n-equations with n-unknowns $c_1, c_2, \ldots, c_n$. It has a unique sol$^{\underline{n}}$ (no matter what $\phi(x_0), \ldots, \phi^{(n-1)}(x_0)$ might be) provided the matrix A has $\det(A) \neq 0$.

$$\det(A) = \begin{vmatrix} Y_1(x_0) & Y_2(x_0) & \cdots & Y_n(x_0) \\ \vdots & & & \vdots \\ Y_1^{(n-1)}(x_0) & Y_2^{(n-1)}(x_0) & \cdots & Y_n^{(n-1)}(x_0) \end{vmatrix} \neq 0$$

(we're using a well known fact from linear algebra here)

So we can conclude that if $Y_1, Y_2, \ldots, Y_n$ satisfy this condition they will construct the general sol$^{\underline{n}}$ to $L[Y] = 0$.

Def$^{\underline{n}}$/ $f_1, f_2, \ldots, f_n$ be n-functions which are at least $(n-1)$-times differentiable. Then

$$W[f_1, f_2, \ldots, f_n](x) = \begin{vmatrix} f_1(x) & \cdots & f_n(x) \\ \vdots & & \vdots \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

Wronskian of $f_1, f_2, \ldots, f_n$

Th^m/ Let $Y_1, Y_2, \ldots, Y_n$ be n-sol^n's on (a,b) of

$$\boxed{Y^{(n)}(x) + P_1(x)\,Y^{(n-1)}(x) + \cdots + P_n(x)\,Y(x) = 0} \quad (*)$$

where $P_i(x)$ are continous on (a,b). If at some point $x_0 \in (a,b)$ these sol^n's have a non-zero Wronskian

$$W[Y_1, Y_2, \ldots, Y_n](x_0) \neq 0$$

then every sol^n of $(*)$ can be expressed in the form

$$Y(x) = C_1 Y_1(x) + C_2 Y_2(x) + \cdots + C_n Y_n(x)$$

- This theorem summarizes what we learned from the proposition, namely that there are at most n-"different" sol^n's to an n^{th} order homogeneous ODE. But what do I really mean by different?

Def^n/ The n-functions $f_1, f_2, \ldots, f_n$ are <u>linearly independent</u> on an interval I iff

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \Rightarrow c_1 = c_2 = \cdots = c_n = 0.$$

otherwise we say $f_1, f_2, \ldots, f_n$ are linearly dependent.

<u>Examples</u>

$$\left.\begin{aligned} f_1(x) &= \sin(x) \\ f_2(x) &= \cos(x) \end{aligned}\right\} \quad \begin{aligned} &c_1 \sin(x) + c_2 \sin(x) = 0 \\ &\Rightarrow c_1 = 0 \text{ and } c_2 = 0. \end{aligned}$$

$$\left.\begin{aligned} f_1(x) &= x \\ f_2(x) &= 5x \end{aligned}\right\} \quad \begin{aligned} &c_1 x + c_2 5x = 0 \\ &\Rightarrow (c_1 + 5c_2)x = 0 \\ &\Rightarrow c_1 = -5c_2 \quad \text{(need not be zero!)} \end{aligned}$$

<u>Remark</u>: If two functions are linearly dependent then they are the same sol^n for most intents and purposes (modulo IVPs where $c_1, c_2, \ldots$ are fixed)

<u>Remark</u>: If $f_1, f_2, \ldots, f_n$ are sol^n's to some homogeneous n^{th} order ODE then $W[f_1, \ldots, f_n](x) \neq 0 \Rightarrow$ they are linearly independent. <u>HOWEVER</u> this only works for <u>COMMON SOLUTIONS</u>, if $g_1, g_2, \ldots, g_n$ are just n-functions then the Wronskian tells us nothing about linear independence. (We'll encounter this in §6.4 with "$W_k(x)$")

Linearly Independent Functions

$\{1, x, x^2, \dots, x^n\}$
$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx\}$
$\{e^{\alpha_1 x}, \dots, e^{\alpha_n x}\}$

These sets are linearly independent. I borrowed those from the text, I think it's also good to know more sets of LI functions

$\{x, x\sin x, xe^x, x^2e^x\}$
$\{\cosh(x), \cos(x), \sinh(x), \sin(x)\}$

the examples are endless really. One thing we should pause and consider is that <u>linear independent functions allow us to equate coefficients</u>. For example,

$$15\sin(x) + 3B\cos(x) + 7 = (A+B)\sin(x) + C$$

yields three equations,

$$\left.\begin{aligned} 15 &= A+B \\ 3B &= 0 \\ 7 &= C \end{aligned}\right\} \text{equating coeff. of the functions} \left\{\begin{aligned} &\sin(x) \\ &\cos(x) \\ &1 \end{aligned}\right.$$

We'll use this idea <u>many</u> times. It is an essential algebraic notion that rests on the linear independence of functions involved if you think about it.

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Thm (Nonhomogeneous Linear ODEs). Given $P_1, P_2, \dots, P_n, g$ contin. on (a,b),

$$\boxed{y^{(n)}(x) + P_1(x)\,y^{(n-1)}(x) + \dots + P_n(x)y(x) = g(x)} \quad (\star)$$

and $\{y_1, y_2, \dots, y_n\}$ the fundamental sol^n set to the auxillary eq^n

$$y^{(n)}(x) + P_1(x)\,y^{(n-1)}(x) + \dots + P_n(x)\,y(x) = 0$$

We may form any sol^n on (a,b) in the form

$$y(x) = y_p(x) + C_1y_1(x) + \dots + C_ny_n(x) \qquad \text{(general sol}^n\text{)}$$

where y_p is a sol^n to $(\star)$ which we call the <u>particular</u> sol^n.

<u>Remark:</u> We can solve nonhomogeneous DEq^n's by first solving the corresponding auxillary DEq^n then simply add y_p to that. The real trouble is finding y_p.

DIFFERENTIAL OPERATORS

The differential operator $D = \frac{d}{dx}$ has nice properties

1.) $D[fg] = D[f]\cdot g + f \cdot D[g]$

I mean multiplication of functions here.

2.) $D[x] = 1$

3.) D is a linear operator.

So we've seen that $e^{\lambda x}$ solves $(D-\lambda)Y = 0$ but what about $(D-\lambda)^2 Y = 0$ where the squared means composition of operators $(D-\lambda)\circ(D-\lambda)Y = 0$. Consider then that $e^{\lambda x}$ is certainly one sol^{n} since,

$$(D-\lambda)^2 Y = (D-\lambda)(D-\lambda)Y = (D-\lambda)(0) = 0.$$

$$\boxed{\text{Claim: } Y = xe^{\lambda x} \text{ is a } 2^{nd} \text{ sol}^{n} \text{ to } (D-\lambda)^2 Y = 0.}$$

__Proof:__ Will use properties 1. and 2. of D,

$$\begin{aligned}
(D-\lambda)^2 xe^{\lambda x} &= (D-\lambda)\left(D(xe^{\lambda x}) - \lambda xe^{\lambda x}\right) \\
&= (D-\lambda)\left((Dx)\cdot e^{\lambda x} + x\cdot(De^{\lambda x}) - \lambda xe^{\lambda x}\right) \\
&= (D-\lambda)\left(e^{\lambda x} + \cancel{x\lambda e^{\lambda x}} - \cancel{\lambda xe^{\lambda x}}\right) \\
&= (D-\lambda)\,e^{\lambda x} \\
&= 0
\end{aligned}$$

And it's not hard to see that this works for $\lambda \in \mathbb{C}$.

$$\boxed{\text{Claim: } Y = x^2 e^{\lambda x} \text{ is a sol}^{n} \text{ to } (D-\lambda)^3 Y = 0.}$$

__Proof:__ Use what we already know

$$\begin{aligned}
(D-\lambda)^3 x^2 e^{\lambda x} &= (D-\lambda)^2\left((Dx^2)e^{\lambda x} + \cancel{x^2\cdot De^{\lambda x}} - \cancel{\lambda x^2 e^{\lambda x}}\right) \\
&= (D-\lambda)^2\left(2xe^{\lambda x}\right) \\
&= 2(D-\lambda)^2(xe^{\lambda x}) \\
&= 0 \qquad \text{by our previous claim.}
\end{aligned}$$

$$\boxed{\begin{gathered}\text{Claim: } (D-\lambda)^n Y \text{ has } n\text{-sol}^{n}\text{'s they are} \\ Y_1 = e^{\lambda x},\ Y_2 = xe^{\lambda x},\ \dots,\ Y_n = x^{n-1}e^{\lambda x}\end{gathered}}$$

__Remark:__ These could be complex if λ were complex, we will see how to extract **real** sol^{n}'s from the complex sol^{n}'s soon.

USEFUL FACTS ABOUT OPERATORS AND COMPLEX NUMBERS

Consider again the operator $L = D - \lambda$. This operator acts on functions of x as follows,

$$L[Y] = DY - \lambda Y$$
$$L[Y](x) = Y'(x) - \lambda Y$$

What function satisfies $L[Y] = 0$? I claim that $e^{\lambda x}$ will do the trick,

$$L[e^{\lambda x}](x) = \frac{d}{dx}\left(e^{\lambda x}\right) - \lambda e^{\lambda x} = \lambda e^{\lambda x} - \lambda e^{\lambda x} = 0.$$

This calculation certainly makes sense if $\lambda \in \mathbb{R}$, but in fact it also is reasonable for $\lambda \in \mathbb{C}$. We pause to discuss the meaning of $e^{\lambda x}$ when $\lambda \in \mathbb{C}$.

- We always assume x is a real variable.

Def^{n}/ Let $a, b \in \mathbb{R}$ and $a+ib \in \mathbb{C}$ where $i = \sqrt{-1}$

$$e^{a+ib} \equiv e^{a}e^{ib} \quad \text{where } e^{ib} \equiv \cos b + i \sin b$$

these eq^{n}'s define what is meant by the exponential of a complex number

TERMINOLOGY: Functions which map $\mathbb{R} \longrightarrow \mathbb{C}$ are complex-valued functions of a real variable. For example,

$$\tilde{f}(x) = a(x) + i\,b(x)$$

Here a, b are functions from $\mathbb{R} \rightarrow \mathbb{R}$, I use the $\sim$ to emphasize $\tilde{f}(x)$ is complex.

- DIFFERENTIATING COMPLEX-VALUED FUNCTIONS

$$\frac{d}{dx}\left(\tilde{f}(x)\right) = \frac{d}{dx}\left(a(x) + ib(x)\right) = \frac{da}{dx} + i\frac{db}{dx}$$
$$\tilde{f}'(x) = a'(x) + i\,b'(x)$$

Just like differentiating $\vec{r}(t)$ from ma 242, we differentiate each component function $(\vec{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle.)$

COMPLEX VALUED FUNCTIONS OF A REAL VARIABLE

Def^{n}/ $\tilde{f}(x) = a(x) + ib(x)$ has real and imaginary components which are both real-valued functions

$$\mathrm{Re}\{\tilde{f}\} = a \quad \text{and} \quad \mathrm{Im}\{\tilde{f}\} = b$$

Now lets go back and verify my claim that $(D-\lambda)e^{\lambda x} = 0$ even when $\lambda \in \mathbb{C}$. Let $\lambda = a+ib$ notice $\mathrm{Re}\{\lambda\} = a$ and $\mathrm{Im}\{\lambda\} = b$,

$$\begin{aligned}
(D-\lambda)e^{\lambda x} &= \left(\frac{d}{dx} - \lambda\right)e^{(a+ib)x} \\
&= \frac{d}{dx}\left[e^{ax}(\cos bx + i\sin bx)\right] - \lambda e^{\lambda x} \\
&= ae^{ax}(\cos bx + i\sin bx) + e^{ax}(-b\sin bx + bi\cos bx) - \lambda e^{\lambda x} \\
&= e^{ax}(a\cos bx + ia\sin bx - b\sin bx + ib\cos bx) - \lambda e^{\lambda x} \\
&= (a+ib)e^{ax}(\cos bx + i\sin bx) - \lambda e^{\lambda x} \\
&= \lambda e^{\lambda x} - \lambda e^{\lambda x} = 0 \qquad \underline{\text{as claimed.}}
\end{aligned}$$

Used $-1 = i \cdot i$

Remark. If you take ma 513 you'll learn how to differentiate w.r.t. a complex variable, complex numbers for us are just an algebraic convenience we will ultimately be interested in real-valued functions of a real variable.

FACT: If $L[\tilde{y}](x) = 0$ then the real and imaginary parts are sol^{n}'s as well; $L[\mathrm{Re}\{\tilde{y}\}](x) = 0$ and $L[\mathrm{Im}\{\tilde{y}\}](x) = 0$. I've assumed that L is a linear operator with property

$$L[\tilde{c}_1 y_1 + \tilde{c}_2 y_2] = \tilde{c}_1 L[y_1] + \tilde{c}_2 L[y_2]$$

for $\tilde{c}_1, \tilde{c}_2 \in \mathbb{C}$. When $L = D^n + P_1 D^{n-1} + \cdots + P_n$ we certainly have this property. (L is "complex linear")

Algebra Refresher

Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial with real coefficients $a_{n-1}, \dots, a_2, a_1, a_0$ then the fundamental Th^{m} of algebra says $\exists$ n-roots say $\lambda_1, \lambda_2, \dots, \lambda_n$ that satisfy $P(\lambda_i) = 0$. These may be repeated and/or complex. This means we can factor

$$P(x) = (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}\cdots(x-\lambda_d)^{m_d}$$

Where $m_1 + m_2 + \cdots + m_d = n$. Now in the event some $\lambda_k = a+ib$ is a sol^{n} it follows that $\lambda_k^* = a - ib$ is likewise a sol^{n}. That is complex roots come in conjugate pairs. This is a straight-forward consequence of the assumption the coefficients are real, if we didn't have the conjugate root then the factorization once multiplied out would give non-real coeff. on the other hand notice $(a, b \in \mathbb{R})$

$$\begin{aligned}(x-(a+ib))(x-(a-ib)) &= x^2 - x(a-ib) - (a+ib)x + (a+ib)(a-ib) \\ &= x^2 - 2ax + \cancel{ibx} - \cancel{ibx} + a^2 - \cancel{iba} + \cancel{iba} - i^2b \\ &= \underbrace{x^2 - 2ax + a^2 + b^2}_{\text{an irreducible quadratic.}}\end{aligned}$$

This is the beauty of conjugate pairs, they make the i factors cancel out. Anyway this means that we can factor *__any__ polynomial with __real__ coefficients into a bunch of real linear factors and irred. quad. factors (possibly repeated)*

$$P(x) = \underbrace{(x-\lambda_1)^{m_1}\cdots(x-\lambda_r)^{m_r}}_{\lambda_1,\dots,\lambda_r \in \mathbb{R}} \underbrace{(x^2+B_1x+C_1)^{n_1}\cdots(x^2+B_sx+C_s)^{n_s}}_{\substack{B_1^2-4C_1, \dots, B_s^2-4C_2 < 0 \\ \text{irreducible quad. factors.}}}$$

With $n = m_1 + \cdots + m_r + 2n_1 + \cdots + 2n_s$.

- *This is the factorization that most explicitly reveals the aspects of $P(x)$ we need to use.*

FIND REAL sol^{n}'s to $(D-\lambda)^n(D-\lambda^*)^n Y = 0$

$\lambda = a+ib$
$\lambda^* = a-ib$

We know that $(D-\lambda)^n Y = 0$ has sol^{n}'s
$\tilde{Y}_1 = e^{\lambda x}$, $\tilde{Y}_2 = xe^{\lambda x}$, $\dots$, $\tilde{Y}_n = x^{n-1}e^{\lambda x}$
but $\lambda = a+ib$ so these are complex,

$$\tilde{Y}_1 = e^{ax}\cos bx + i e^{ax}\sin bx$$
$$\tilde{Y}_2 = xe^{ax}\cos bx + ixe^{ax}\sin bx$$
$$\vdots$$
$$\tilde{Y}_n = x^{n-1}e^{ax}\cos bx + i x^{n-1}e^{ax}\sin bx$$

Notice $(D-\lambda^*)^n Y^*$ likewise has sol^{n}'s with $\lambda^* = a-ib$

$$\tilde{Y}_1^* = e^{ax}\cos bx - i e^{ax}\sin bx$$
$$\vdots$$
$$\tilde{Y}_n = x^{n-1}e^{ax}\cos bx - i x^{n-1}e^{ax}\sin bx$$

This means that $(D-\lambda)^n$ and $(D-\lambda^*)^n$ both possess real sol^{n}'s

$$\boxed{\{e^{ax}\cos bx,\ e^{ax}\sin bx,\ \dots,\ x^{n-1}e^{ax}\cos bx,\ x^{n-1}e^{ax}\sin bx\}}$$

of which we can count $2n$-sol^{n}'s in total.

<u>Summary</u>: we have found how to assemble real sol^{n}'s in the complex case and we already saw how to find sol^{n}'s in the real case with repeats. So we now have all we need to detail the general sol^{n} to.

$$Y^{(n)}(x) + P_1(x)Y^{(n-1)}(x) + \dots + P_n(x)Y(x) = 0$$

- Although to be careful we ought to prove these sol^{n}'s we have found are indeed linearly indenpendent. You should know how to prove LI for specific examples, you could use the Wronskian to check LI since these functions are all sol^{n}'s to a common DEq^{n}.

Solving n^{th} order constant coefficient linear ODEs

We wish to solve the following DEq^{n}:

$$\boxed{y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_{n-1} y'(x) + a_n y(x) = 0} \quad \text{Eq}^{n}(1)$$

Here we assume $a_1, \ldots, a_{n-1}, a_n$ are real constants which certainly are continous on all of $\mathbb{R}$ so once we have n-intial conditions we know by existence/uniqueness Th^{m} the sol^{n} will exist and be defined for all $\mathbb{R}$. So let's find it, recast Eq^{n}(1) as an operator eq^{n}

$$\boxed{\left(D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n\right) y \equiv L[y] = 0} \quad \text{Eq}^{n}(2)$$

Notice that the linear operator L is a polynomial in the operator $D \equiv \frac{d}{dx}$.

$$\boxed{L = P(D) = D^n + a_1 D^{n-1} + \cdots + a_n}$$

This polynomial P has real coefficients $a_1, \ldots, a_n$ so we can factor it as discussed in ALGEBRA REFRESHER

$$0 = L[y] = (D-\lambda_1)^{m_1} \cdots (D-\lambda_r)^{m_r} (D^2 + B_1 D + C_1)^{n_1} \cdots (D^2 + B_s D + C_s)^{n_s} y$$

Solving this eq^{n} is easy with what we know. Notice that all we need for y to be a sol^{n} is that one of the factors annihilate it. In total we get n-LI sol^{n}'s from the various factors

linear factors $\quad (D-\lambda_1)^{m_1} \Rightarrow y_1 = e^{\lambda_1 x}, \ldots, y_{m_1} = x^{m_1 - 1} e^{\lambda_1 x}$

irreducible quad. factors $\quad (D^2 - B_1 D + C_1)^{n_1} = (D - (\alpha + i\beta))^{n_1} (D - (\alpha - i\beta))^{n_1}$

$$\Rightarrow y_{M+1} = e^{\alpha x} \cos \beta_1 x, \quad y_{M+2} = e^{\alpha x} \sin \beta_1 x$$
$$y_{M+3} = x e^{\alpha x} \cos \beta_2 x, \quad y_{M+4} = x e^{\alpha_2 x} \sin \beta_2 x, \ldots$$

Where I made up $M \equiv m_1 + m_2 + \cdots + m_r$ to keep the labeling correct. So we find in total n- LI sol^{n}'s to Eq^{n}(1), we'll write the general sol^{n} in all it's glory on the next page ↻

Let us write the general sol^{n} to Eq^{n} (1),

$$\boxed{\begin{aligned} y = {} & c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x} + \cdots + c_{m_1} x^{m_1 - 1} e^{\lambda_1 x} + \cdots \\ & + c_{M - m_r} e^{\lambda_{m_r} x} + \cdots + c_M x^{m_r - 1} e^{\lambda_{m_r} x} + \\ & + c_{M+1} e^{\alpha_1 x} \cos \beta_1 x + c_{M+2} e^{\alpha_1 x} \sin \beta_1 x + \cdots \\ & + c_{M + 2n_1 - 1} x^{n_1 - 1} e^{\alpha_1 x} \cos \beta_1 x + c_{M + 2n_1} x^{n_1 - 1} e^{\alpha_1 x} \sin \beta_1 x + \cdots \\ & + c_{n - 2n_s} e^{\alpha_s x} \cos \beta_s x + c_{n - 2n_s + 1} e^{\alpha_s x} \sin \beta_s x + \cdots \\ & + c_{n-1} x^{n_s - 1} e^{\alpha_s x} \cos \beta_s x + c_n x^{n_s - 1} e^{\alpha_s x} \sin \beta_s x \end{aligned}}$$

Remark: I don't remember this formula, instead I use the principles that led us to it to assemble sol^{n}'s for particular examples.

Remark: there are ∞ly many possible sol^{n}'s until we are supplied the needed n- initial conditions.

Auxillary Eq^{n}:

We have used operator arguments to assemble this sol^{n} but we could just as well supposed that $e^{\lambda x}$ was a sol^{n} to find

$$L[e^{\lambda x}] = P(\lambda)\, e^{\lambda x}$$

Where this is the same polynomial we found in D. Clearly $L[e^{\lambda x}] = 0$ iff $P(\lambda) = 0$. In practice I prefer to factor the auxillary eq^{n} in terms of λ as opposed to a polynomial in D.

• (The text uses "r" instead of "λ" so be warned.) •

• I'll factor $P(\lambda)$ to find $\lambda_1, \lambda_2, \ldots, \lambda_r$ and $\alpha_1, \beta_1, \ldots, \alpha_s, \beta_s$ and the multiplicities.

Examples

E1 $y'''(x) = 0$

$\lambda^3 = 0 \Rightarrow \boxed{y = c_1 + c_2 x + c_3 x^2}$ (since $e^{0\cdot x} = 1$)

E2 $y'' + y = 0$

$\lambda^2 + 1 = 0$

$\Rightarrow \lambda = \pm i = \alpha \pm i\beta$

identify $\alpha = 0$ & $\beta = 1$ $\therefore$ $\boxed{y = c_1 \cos(x) + c_2 \sin(x)}$

E3 $y''' + 3y'' + 3y' + 3y = 0$

$\lambda^3 + 3\lambda^2 + 3\lambda + 3 = 0$

$(\lambda + 1)^3 = 0$

$\lambda = -1$ with multiplicity 3. $\boxed{y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}}$

E4 $y'''' + 2y'' + y = 0$

$\lambda^4 + 2\lambda^2 + 1 = 0$

$(\lambda^2 + 1)^2 = 0$

$\lambda = \pm i$ with multiplicity 2.

$\therefore$ $\boxed{y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x}$

E5 $y''' + 5y'' + 6y' = 0$

$\lambda^3 + 5\lambda^2 + 6\lambda = 0$

$\lambda(\lambda^2 + 5\lambda + 6) = \lambda(\lambda + 3)(\lambda + 2) = 0$

$\Rightarrow \lambda_1 = 0, \lambda_2 = -3, \lambda_3 = -2$

$\therefore$ $\boxed{y = c_1 + c_2 e^{-3x} + c_3 e^{-2x}}$

E6 $y' = y$

$\lambda = 1$ $\therefore$ $\boxed{y = c_1 e^x}$

E7 $y'' - y = 0$

$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \Rightarrow y = c_1 e^x + c_2 e^{-x}$

Remark: Consider what we've done, we've changed the problem of unraveling a complicated differential eq^n to the relatively easy problem of factoring a polynomial. Algebra has replaced calculus here, it's really quite amazing that constant coefficient linear ODEs are easy