Energy Methods (§ 12.4)

We saw that in $\boxed{E6}$, $\boxed{E7}$, $\boxed{E8}$ the 2^{nd} order DEq^{n} which is Newton's 2^{nd} Law in one-dimension could be converted to a system of two first order DEq^{n}'s. We now continue that discussion. To begin we'll see how to graph sol^{n}'s for <u>conservative forces</u> then we'll tweak the conservative analysis to gain insight into <u>nonconservative forces</u>.

Conservative Systems

By definition if F is a conservative force then there exists a potential U such that $F = -\frac{dU}{dx}$ thus Newton's Law $F = ma$ becomes

$$-\frac{dU}{dx} = m\frac{d^2x}{dt^2} \Rightarrow m\frac{d^2x}{dt^2} + \frac{dU}{dx} = 0$$

Now U is the potential energy and $\frac{1}{2}mv^2$ is the kinetic energy classically. The total energy $E = \frac{1}{2}mv^2 + U$. We can show E is <u>conserved</u> for a conservative system, I give an indirect argument well-known among physicists,

$$m\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dx}{dt}\frac{dU}{dt} = 0 \quad : \text{ multiplied Newton's Law by } \frac{dx}{dt}$$

$$\|$$

$$\frac{d}{dt}\left(\frac{1}{2}m\left(\frac{dx}{dt}\right)^2\right) + \frac{d}{dt}(U) = 0 \quad : \text{ used chain rule assuming } U = U(x).$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}mv^2 + U\right) = 0 \quad : \ v \equiv \frac{dx}{dt}$$

So we find $\frac{dE}{dt} = 0$ <u>when</u> Newton's Law holds. This means that sol^{n}'s of Newton's Eq^{n} will be level curves $E = k$ for conservative systems. We follow the text in what follows,

SOLUTIONS ARE LEVEL CURVES OF ENERGY

Well almost, we divide $mx'' + \frac{dU}{dx} = 0$ by m to obtain,

$$\frac{d^2x}{dt^2} + \frac{d}{dx}\left(\frac{U}{m}\right) = 0 \Rightarrow \boxed{\frac{d^2x}{dt^2} + g(x) = 0} \leftarrow \text{starting point for §12.4's view.}$$

where we define $g(x) = \frac{1}{m}\frac{dU}{dx}$. Let $v = \frac{dx}{dt}$ as usual then we have two first order DEq$^{\underline{n}}$'s in x & v,

$$\boxed{\frac{dx}{dt} = v \quad \text{and} \quad \frac{dv}{dt} = -g}$$

by the same calculation as before we can argue E is conserved, we define $G \equiv \int g(x)dx$ the potential function ($G = U/m$) and the energy function (actually its E/m)

$$\boxed{E(x,v) \equiv \frac{1}{2}v^2 + G(x)}$$

The eq$^{\underline{n}}$'s of motion are level curves of $E(x,v)$, that is they are curves of the form $E(x,v) = \text{constant}$.

- Customarily we force $E(0,0) = 0$ but this need not be enforced, we can trade G for $G + \text{constant}$ w/o changing the physics

GOAL: Sketch Trajectories in the xv-plane for conservative systems. We proceed by example to begin.

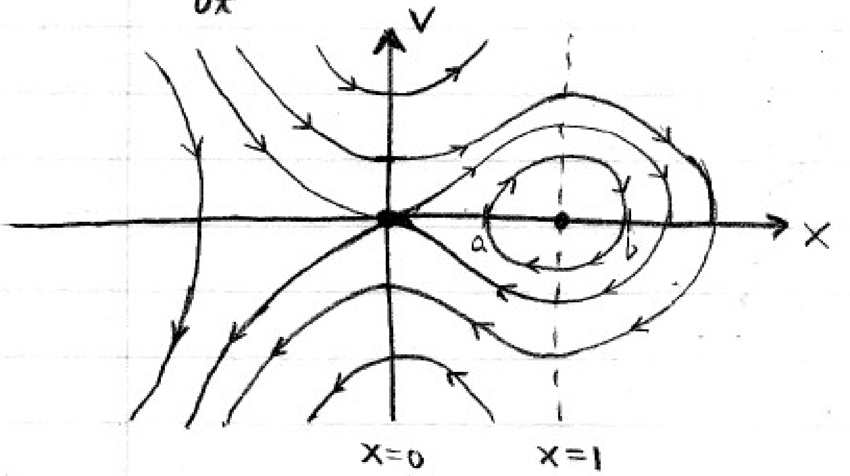
$\boxed{E17}$ Consider the conservative system with

$$\frac{d^2x}{dt^2} + x(x-1) = 0$$

identify $g(x) = x(x-1)$ we study $\frac{dv}{dt} = -x(x-1)$ and $v = \frac{dx}{dt}$. Critical points in the xv-plane satisfy

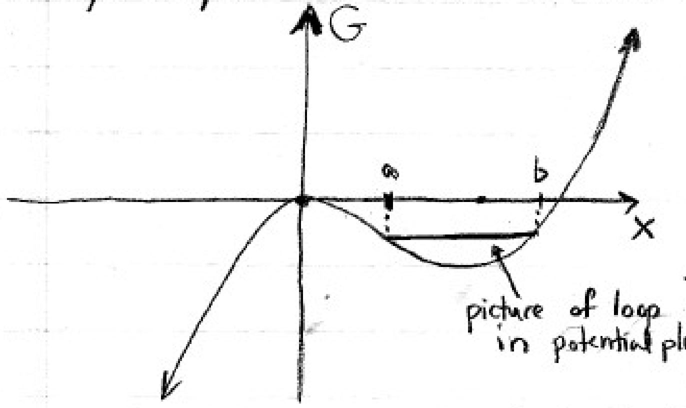
$$\frac{dx}{dt} = v = 0 \qquad \underline{\text{AND}} \qquad \frac{dv}{dt} = -x(x-1) = 0 \begin{array}{l} \nearrow x = 0 \\ \searrow x = 1 \end{array}$$

Thus there are two critical points; $(0,0)$ and $(1,0)$. Lets see we know sol^{n}'s flow to the right for $v > 0$ and to the left for $v < 0$ because $dx/dt = v$. Also we can see $\frac{dv}{dt} > 0$ for $0 < x < 1$, whereas $\frac{dv}{dt} < 0$ for $x < 0$ or $x > 1$.



- also $\frac{dv}{dt} = 0$ at $x=0$ or $x=1$. those are horiz. tangents
- when $\frac{dx}{dt} = 0$ get vertical tangents (that is at $v=0$)

The phase plane plot above follows from analyzing $x' = v$ & $v' = -x(x-1)$ directly. However, the better way to do it is to use the potential G to anticipate the phase plane. Notice how min/max's line up with critical points.



$$g(x) = x(x-1) \Rightarrow G(x) = \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2$$

General Comments, how to graph $E = \text{constant}$ with the help of the potential function $G = \int g(x)\,dx$

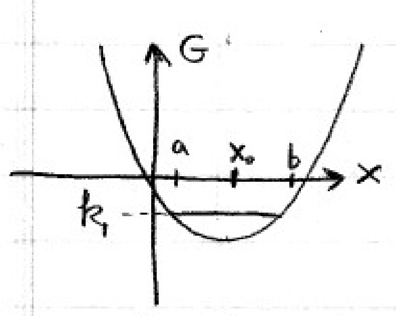
We study $E(x,v) = k$ where k is a constant. That is

$$\tfrac{1}{2}v^2 + G = k$$

$$v^2 = 2(k - G)$$

since $v^2 \geq 0$ we will only find a sol^n curve for $k - G \geq 0$ given $k - G \geq 0$ we have $v = \pm\sqrt{2(k-G)}$ which gives three types of behaviour about critical points. Notice that if $\frac{dx}{dt} = v$ and $\frac{dv}{dt} = -g$ where $G = \int g\,dx$ our critical point in xv-plane has $v = 0$ and $-g = 0$. Notice since $\frac{dG}{dx} = \frac{d}{dx}\int g\,dx = g$ we find $\frac{dG}{dx} = 0$ at the critical point, therefore the critical point in xv-plane corresponds to a critical point for G in the sense of calc I. Suppose $(x_0, 0)$ is a critical point then level curves $E = k$ near $(x_0, 0)$ follow from the nature of $G(x_0)$, is it a local min/max and is k close to $G(x_0)$? Proceed case by case,

Local Minimum



$G = k_1$ at $x = a$ and $x = b$ that is $k_1 = G(a) = G(b)$ and $G(x_0) < k_1$

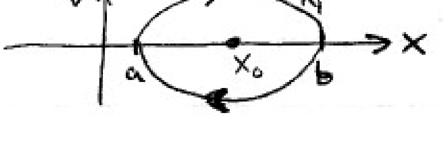
in the case considered here

$k_1 - G \geq 0$

for each x in $a \leq x \leq b$

$v = \sqrt{2(k_1 - G)}$ & $v = -\sqrt{2(k_1 - G)}$

join together at $x = a$ and $x = b$ where $k_1 = G$ so $v = 0$. This makes some sort of circle-like thing

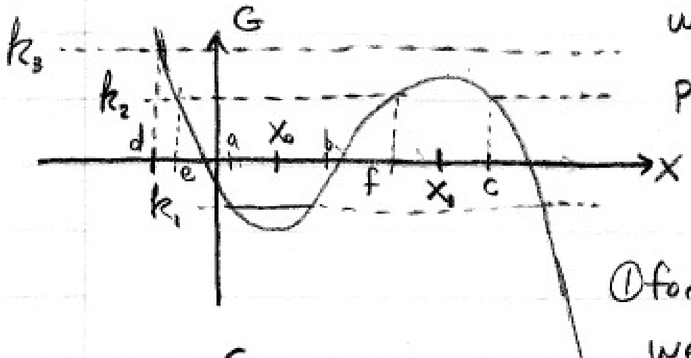


LOCAL MINIMUM CONTINUED

If $G(x_0) < k$ and k is less than any nearby local maximums ($G(x_1) > k$ if $G(x_1)$ is a local max near x_0) then we find the situation just described. If $G(x_0) > k$ then $k - G < 0$ so no real sol$^{\underline{n}}$s to $v = \pm\sqrt{2(k-G)}$ are found. When $k = G(x_1) > G(x_0)$ that brings us to the next case.

LOCAL MAXIMUM'S FOR G

Essentially we consider $G(x_1)$ a local max. that looks something like the following ($G'(x_1) = 0$, $G''(x_1) < 0$)

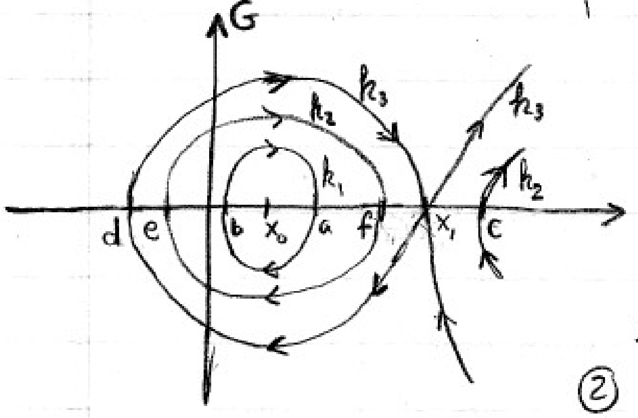


we'll keep the min at $G(x_0)$ for perspective. Observe

① $k_3 > G(x_1)$

② $k_2 < G(x_1)$

① for k_3 since $k_3 - G(x) > 0$ we find level curves $v = \sqrt{2(k_3 - G)}$ and $v = -\sqrt{2(k_3 - G)}$. It appears at $x = d < 0$ these curves meet since $k_3 = G(d)$. However, for $x > x_1$ there doesn't appear to be such a point.



② for k_2 we see $k_2 - G(x) \geq 0$ only for $x \geq c$ or for $e \leq x \leq f$ this gives the unbounded curve through $x = c$ and the loop that closes on $x = e$ & $x = f$.

Finally $k_3 = G(x_1)$ gives the loop around $(x_0, 0)$ which feeds into $(x_1, 0)$. See Figure 12.21 for a similar analysis with nicer pictures.

$\boxed{E18}$ Analyze $\frac{d^2x}{dt^2} + x - x^3 = 0$ using the <u>Energy Analysis</u>

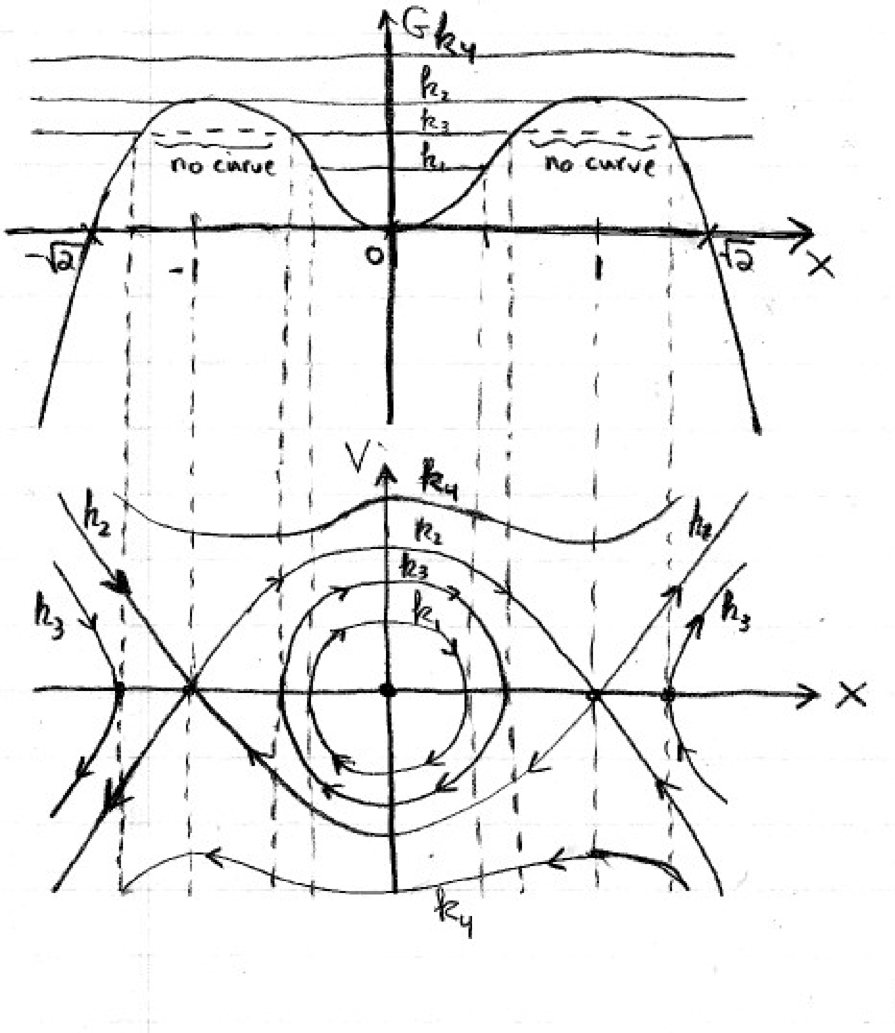
Identify that $g(x) = x - x^3$ thus $G(x) = \frac{1}{2}x^2 - \frac{1}{4}x^4 + C$
we choose $C = 0$ so that $E(x,v) = \frac{1}{2}v^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4$,
this gives $E(0,0) = 0$ as is our custom (if possible)
To begin we graph $G(x)$ carefully with the help of calc I min/max theory.

$$G'(x) = x - x^3 = 0 \Rightarrow x(1-x^2) = 0 : \text{for critical pts.}$$
$$\Rightarrow \underline{x = 0,\ x = 1,\ x = -1 : \text{critical pts.}}$$
$$G''(x) = 1 - 3x^2 \Rightarrow G''(0) = 1,\ G''(1) = -2,\ G''(-1) = -2$$

By 2^{nd} Derivative Test → $G(0)$ is local min
→ $G(1)$ is local max
→ $G(-1)$ is local max

So we find $G(x)$ is a quartic that looks like, also note zeroes of $G(x)$ are sol^{n}'s to

$$\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4 = 0$$
$$\tfrac{1}{2}x^2\left(1 - \tfrac{1}{2}x^2\right) = 0$$
$\therefore$ $x = 0$ and $x^2 = 2$
so $x = 0$, $x = \pm\sqrt{2}$ are roots for G



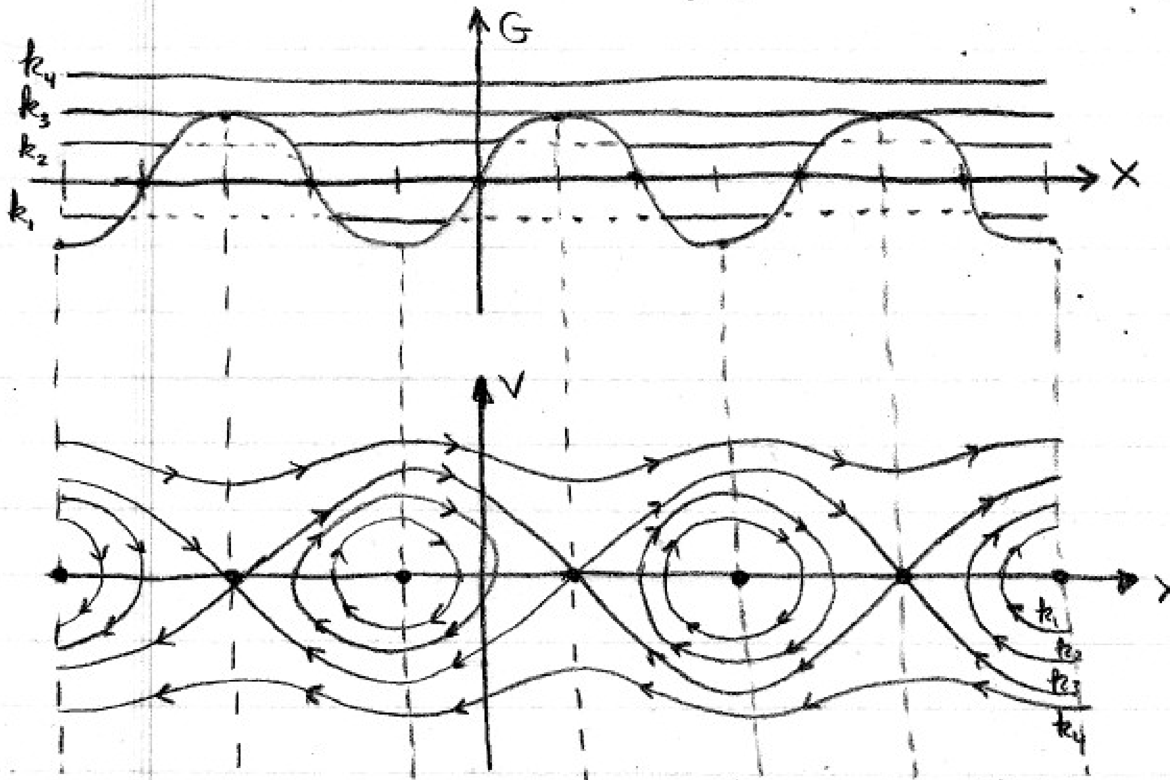
- To draw the phase plane we simply consider how various energy levels are bounded in the xG-plot
- closed trajectories around min
- unstable trajectories about max
- same critical points (in terms of x-coordinate)
- $v = \frac{dx}{dt}$ tells where arrows should be placed.

E19 $\frac{d^2x}{dt^2} + \cos x = 0$ Plot phase plane using energy analysis.

Note $g(x) = \cos(x)$ thus $G(x) = \int \cos(x)dx = \sin(x)$.
Hence $E = \frac{v^2}{2} + \sin(x)$ and as usual we wish to plot trajectories which are easily graphed since each trajectory is a level curve of the energy function. To begin analyze G,

$$G(x) = \sin(x) \Rightarrow G'(x) = \cos(x)$$
$$\Rightarrow G'(x) = 0 = \cos(x) \quad : \text{critical points}$$
$$\Rightarrow x = n\pi + \pi/2 \qquad n = 0, \pm 1, \pm 2, \ldots$$

Of course we know what the graph of $\sin(x)$ is w/o calculus,



$\frac{dx}{dt} = V$ reveals the direction of the trajectories

- we see there are certain bounded motions ($E < k_3$) and other unbounded ($E \geq k_3$)
- As you can see the critical points $x = \frac{-\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$ are stable centers whereas $x = \frac{-3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$ are
- Notice that we just plotted the phase plane for a nonlinear system: $\frac{dx}{dt} = V$, $\frac{dV}{dt} = -\cos(x)$ we didn't even have to think that hard to do it, as an added bonus we have a physical interpretation.

ENERGY ANALYSIS FOR NONCONSERVATIVE SYSTEMS

A nonconservative system often consists of a conservative piece $F = -\frac{dU}{dx}$ and a friction-like term $-mh(x, dx/dt)$ which gives us Newton's 2nd Law,

$$m\frac{d^2x}{dt^2} = -\frac{dU}{dx} - mh(x, dx/dt)$$

Defining $g(x) = \frac{1}{m}\frac{dU}{dx}$ as before and dividing by m yields

$$\boxed{\frac{d^2x}{dt^2} + g(x) + h(x, dx/dt) = 0} \leftarrow (*)$$

we now discuss how to plot the phase plane for this "non conservative" system. Notice the dx/dt dependence inside h is the key distinction from the examples considered thus far. We define the energy function as before, defining $G(x) = \int g(x)dx$,

$$\boxed{E = \frac{1}{2}v^2 + G(x)}$$

In the nonconservative case E is not conserved. Lets see why,

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2}v^2 + G\right]$$

$$= v\frac{dv}{dt} + \frac{dG}{dt}$$

$$= v\frac{dv}{dt} + \frac{dx}{dt}\frac{dG}{dx}$$

$$= v\left(\frac{dv}{dt} + \frac{dG}{dx}\right) \qquad \text{note } \frac{dG}{dx} = g \;\&\; \frac{dv}{dt} = -g - h$$

$$= v(-g - h + g)$$

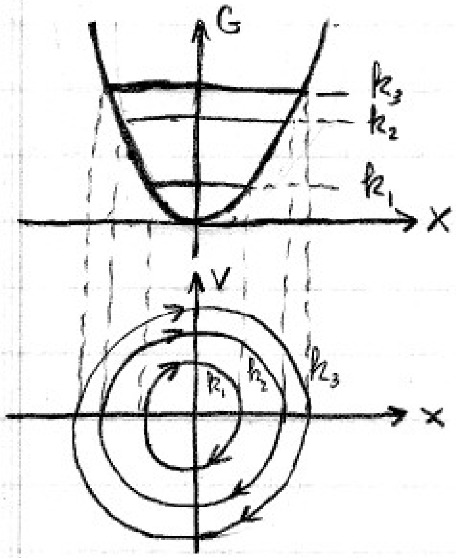
$$\boxed{\frac{dE}{dt} = -vh} \leftarrow \text{the energy changes as given here.}$$

(Energy Change Eq^{n})

Remark: We can take a nonconservative system like (*) and set $h=0$ to obtain a corresponding conservative system. The Energy Change Eq: then shows us how to craft the phase plane plot for the nonconservative system in terms of shifting from one energy level to another of the corresponding conservative system.

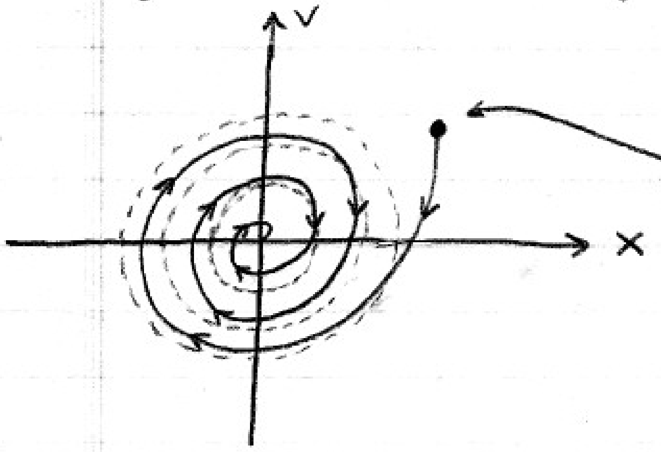
$\boxed{E20}$ Building on $\boxed{E7}$ but change "y" to "x" consider the spring with friction, $\frac{d^2x}{dt^2} + x + \frac{dx}{dt} = 0$. Here we see $g(x) = x$ so $G(x) = \frac{1}{2}x^2$ while $h = \frac{dx}{dt} = v$

Analyze the corresponding conservative system $\frac{d^2x}{dt^2} + x = 0$ to start,



- I used the energy analysis to plot three trajectories with $E = k_1 < k_2 < k_3$, the circles with larger radii have higher energy.

Now consider the Energy Change Eq: $\frac{dE}{dt} = -vh = -v^2 \leq 0$ the energy decreases as time goes onward. I'll plot one typical trajectory, we pick some point to begin say this one ; . The sol^{n} spirals into the origin as the particle loses energy due to the friction term.



Remark: See Example 3 on pg. 772-773 for another example of this idea.