The Phase Plane : A Graphical Approach

We've seen that n^{th} order ODE's are efficiently solved by a little polynomial algebra, we just factor the auxillarly eq$^{\underline{n}}$ and write exponentials and sines/cosines as we discussed. It turns out a similar procedure works for a systems of ODEs, we'll factor the characteristic eq$^{\underline{n}}$ then find eigenvalues & eigenvectors (plus generalized eigenvectors in special cases) then it's a simple task to construct the general sol$^{\underline{n}}$ to a system of ODE's. We will return to that general task in a few lectures but for now we begin by analyzing systems of two ODE's which are <u>autonomous</u>.

$$\boxed{\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{aligned}} \quad (1)$$

refers to the fact f & g are <u>not</u> nontrivial functions of t.

this system's behaviour can be described qualitatively by sketching trajectories in the "phase plane." The phase plane here for (1) is simply the xy-plane. Then an <u>INTEGRAL CURVE</u> is a curve which solves the <u>PHASE PLANE Eq$^{\underline{n}}$</u>

$$\boxed{\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}} \quad (2)$$

the t-dependence is suppressed in (2). Historically Leibniz viewed the problem of calculus as (2) whereas Newton viewed the problem of calculus as (1). I suppose this is simply because Newton had the problem of classical mechanics always in mind, whereas Leibniz was focusing more just on the math.

Introductory remarks on PHASE PLANE ANALYSIS & MORE (§5.4≈)

Now for the most part we deal with $y = f(x)$ and we write formulas like $\int x\,dx = \frac{1}{2}x^2 + c$, you can thank Leibniz for that approach. Newton in contrast was more interested in the parametric view of things where the parameter "t" really was time. I digress a bit, ultimately the approaches are equivalent. For example, $x^2 + y^2 = 1$ is a circle. However, so is $x = \cos t$, $y = \sin t$ for $0 \leq t \leq 2\pi$. There is extra information in $x = \cos t$, $y = \sin t$, these eq$^{\underline{n}}$'s give the circle a natural counter-clockwise direction. No direction is implicit within $x^2 + y^2 = 1$. So, what I'm getting at here is that it's not surprising that (1) has more info than (2). The sol$^{\underline{n}}$ to (1) would be a pair of eq$^{\underline{n}}$'s $(x(t), y(t))$ which we call a <u>TRAJECTORY</u>.

$\boxed{E1}$ $\left.\begin{aligned} \frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= x \end{aligned}\right\}$ this system has the sol$^{\underline{n}}$ $\left.\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}\right\}$ these are a trajectory for the system.

$\boxed{E2}$ $\dfrac{dy}{dx} = \dfrac{x}{-y}$ has sol$^{\underline{n}}$ $x^2 + y^2 = 1$.

this is verified via implicit differentiation,

$$\frac{d}{dx}\left(x^2 + y^2 = 1\right) \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

<u>Remark</u>: $\boxed{E2}$ has the "phase plane" eq$^{\underline{n}}$ for $\boxed{E1}$ both have circles as sol$^{\underline{n}}$'s, but w/o $\boxed{E1}$'s system of ODEs we could not supply a direction.

WHAT'S MY POINT?

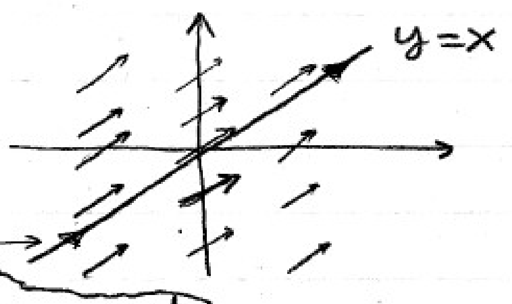
THERE IS AN IMPORTANT CONCEPTUAL DISTINCTION BETWEEN THE DIRECTION FIELDS CONSIDERED IN §1.3 (and also in §7.1–7.2≈ in STEWART'S CALC. & CONCEPTS, CALC II) AND THE PHASE PLANE WE CONSIDER NOW.

DIRECTION FIELDS —— $\frac{dy}{dx} = h(x,y)$ ← need not come from any system.

PHASE PLANE —— $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ ← come's from the system $\frac{dx}{dt} = f,\ \frac{dy}{dt} = g$

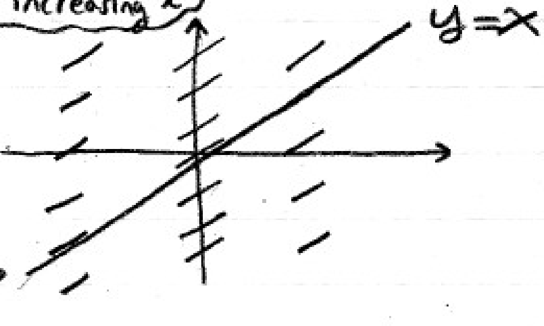
Only for the phase plane do we get to draw arrows on the direction field.

$\boxed{E3}$ $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 1$ ⟶ $\frac{dy}{dx} = 1$



notice the arrow pointing the direction of increasing t

$\boxed{E4}$ $\frac{dy}{dx} = 1$ no sense of direction.



notice here the sol^{n} $y=x$ has no direction

<u>Remark</u>: ok, enough about the distinction, we now move on to the task at hand, how do we analyze the PHASE PLANE Eq^{n}? That is what are the interesting features & how do we find them. The ideas & goals here are much like those of the min/max discussion in Calculus I.

PHASE PLANE, BASIC IDEAS

AGAIN, we consider the autonomous system

$$\left.\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y)\end{aligned}\right\} \longrightarrow \frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

> Def^{n}/ A critical point (x_0, y_0) has $f(x_0, y_0) = 0$ and $g(x_0, y_0) = 0$. The equilibrium sol^{n} is $x = x_0$, $y = y_0$. All such points form the "critical point set".

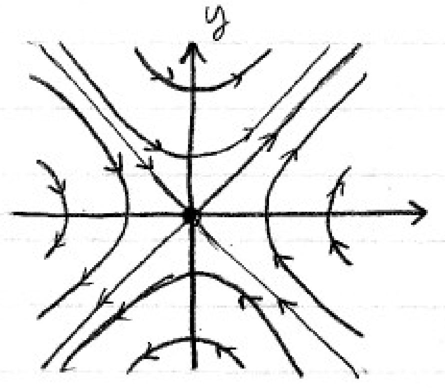
$\boxed{E5}$

$$\left.\begin{aligned}\frac{dx}{dt} &= 2y\\ \frac{dy}{dt} &= 2x\end{aligned}\right\} \longrightarrow \frac{dy}{dx} = \frac{2x}{2y}$$

Solve by seperating variables. (in this example, not always)

$$2y\,dy = 2x\,dx$$

$$y^2 = x^2 + c \quad \therefore \quad y^2 - x^2 = c$$



$$\underbrace{\frac{dx}{dt} = 2y}_{\text{this tells me which way the sol}^{n}\text{'s must flow.}} \Rightarrow$$

sol^{n}'s go to the right for $y > 0$ and to the left for $y < 0$

Critical Point?

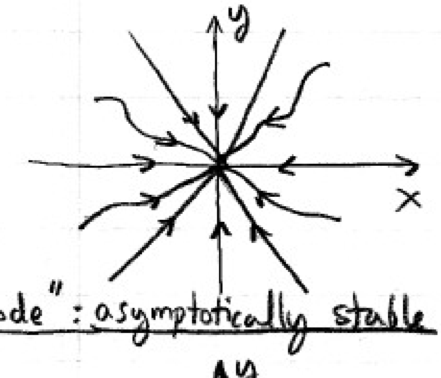
$$\left.\begin{aligned}\frac{dx}{dt} &= 2y = 0\\ \frac{dy}{dt} &= 2x = 0\end{aligned}\right\} \Rightarrow x = 0 \ \& \ y = 0 \quad \therefore \boxed{(0,0) \text{ is the only critical point}}$$

- this critical point is called unstable since some trajectories flow off the point. In contrast Example I on p. 267 shows a stable point.

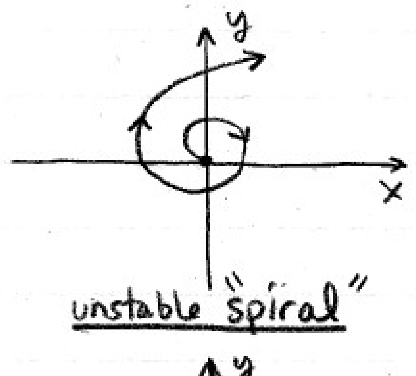
Remark: $(0,0)$ is a saddle point because of $y = -x$ sol^{n}

> Th^{m}/ If a trajectory $(x(t), y(t))$ has a limit point (x^*, y^*) then that limit point is also a critical point. We mean "limit point" to indicate $\lim_{t\to\infty}(x(t), y(t)) = (x^*, y^*)$.

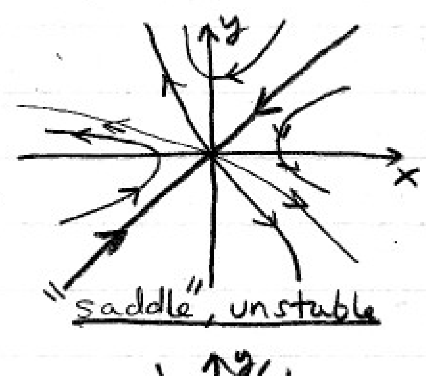
What this means is that either a trajectory will diverge in the sense that $|x(t)| + |y(t)| \longrightarrow \infty$ as $t \to \infty$ or oscilate wildly, but if the trajectory behave's nicely and converges to some point (x^*, y^*) then that point will be a critical point. So we can classify how sol^n's behave as $t \to \infty$ if they converge to some point, in fact we can do a little better, if a sol^n comes close to a critical point we can anticipate it's shape depending on certain criteria which will find out about soon (well... at least for linear or almost linear systems and also systems that allow an energy analysis). The possible limiting behaviours are (following figure 5.12)



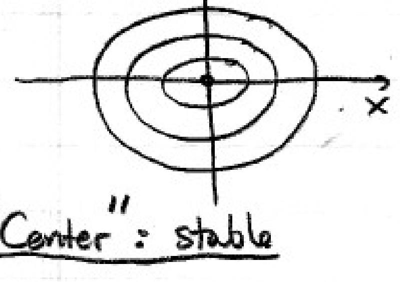
"Node": asymptotically stable



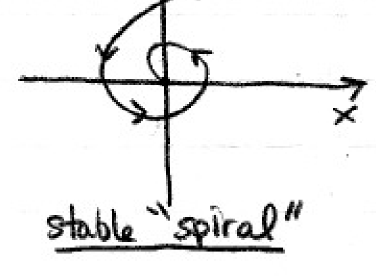
unstable "spiral"



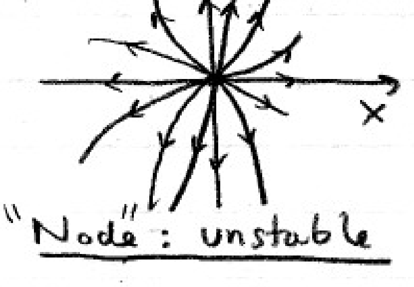
"saddle", unstable



"Center": stable



stable "spiral"



"Node": unstable

Centers are always stable and saddles are always unstable. The others could be either stable or unstable depending on the details of the system. Lets examine a few examples before getting to the detailed systematic analysis.

$\boxed{E6}$ A system of two ODEs can come from a single 2^{nd} order ODE. The most known example is <u>Newton's 2^{nd} Law</u> : $my'' = f(y, y')$. Assuming the net-force is time-independent. If we say $v = dy/dt$ then we have <u>two</u> ODE's instead,

$$y' = v$$
$$v' = f(y, v)/m \qquad (\text{assuming } m \neq 0)$$

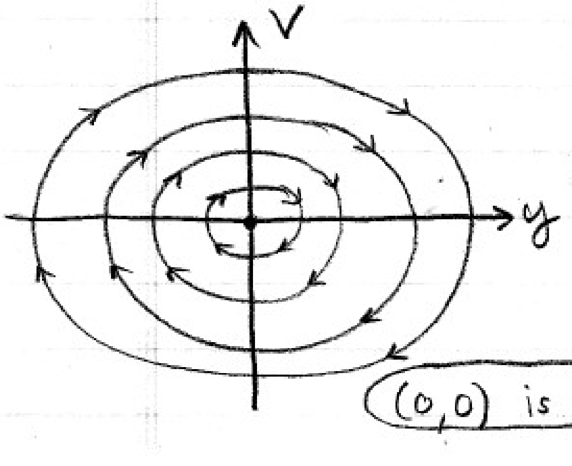
notice we take v and y to be independent variables in this approach, the "phase plane" here is the yv-plane.

$\boxed{E7}$ Consider $f = -ky$ where $k > 0$. This is the eq$^{\underline{n}}$ for a idealized spring $my'' = -ky$. Introducing $v = y'$ we find the system,

$$m\frac{dv}{dt} = -ky \quad \& \quad v = \frac{dy}{dt}$$

using an old but timeless trick $\frac{dv}{dt} = \frac{dy}{dt}\frac{dv}{dy} = v\frac{dv}{dy}$ so,

$$mv\frac{dv}{dy} = -ky \Rightarrow vdv = \frac{-k}{m}ydy$$
$$\Rightarrow v^2 = -\frac{k}{m}y^2 + C$$
$$\Rightarrow \underline{v^2 + \frac{k}{m}y^2 = C}$$



these are ellipses.

- note $v = \frac{dy}{dt} \Rightarrow$ rightward flow for $v > 0$. leftward flow for $v < 0$

(0,0) is critical point, its a center point

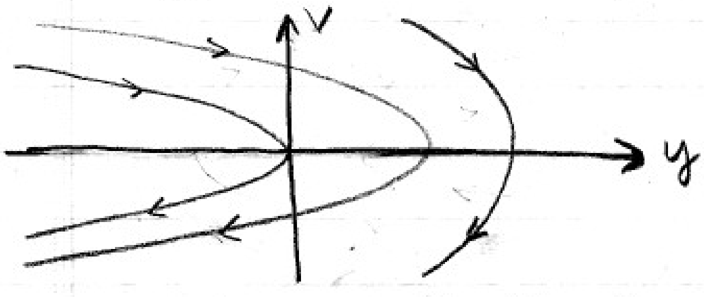
Remark: In $\boxed{E7}$ the ellipses represent possible motions of a spring with mass m. The greater the velocity V at some point in the trajectory the larger the ellipse, hence the larger the oscillation of the spring. This yV-plane or $y\dot{y}$-plane is usually what physicists mean when they mention the "phase plane". By treating velocity as an independent variable we can reduce the 2^{nd} order Newton's Eq$^{=}$ to two first order ODE's, which are equivalent & also tend to reveal important features of the system. Notice, for example, we don't even know the sol$^{=}$'s as functions of time yet we can say with certainty that the motion in y is bounded for all time.

$g = 9.8\,m/s^2$

$\boxed{E8}$ Gravity near surface of earth, $my'' = -mg$ this is kind-of silly but stick with me, $y' = V$ and as before $\frac{dV}{dt} = \frac{dy}{dt}\frac{dV}{dy} = V\frac{dV}{dy}$ thus

$$m\frac{dV}{dt} = -mg \Rightarrow V\frac{dV}{dy} = -g \Rightarrow VdV = -g\,dy$$

$$\Rightarrow \frac{1}{2}V^2 = -gy + C$$



$$\Rightarrow \underline{V^2 = -2gy + C}$$

parabolas open sideways in yV-plane.

(like $y^2 = x$ in usual xy-terms)

here all sol$^{=}$'s eventually flow like $y \to -\infty$ as $t \to \infty$. There is no critical point here since $\frac{dV}{dt} = -g \neq 0$.

Ok, one more example before we start the systematic analysis.

$\boxed{E9}$ Gravity near surface of earth (again). $\boxed{E8}$ suppressed the horizontal motion (that is "x-dependence") ignoring friction & other influences Newton's Second Law gives

$$\left.\begin{aligned} m\ddot{x} &= 0 \\ m\ddot{y} &= -mg \end{aligned}\right\} \text{ system of } 2^{nd} \text{ order ODE's.}$$

for these very simple DEq^{n}'s we can integrate twice to find,

$$x(t) = x_o + V_{ox}t$$
$$y(t) = y_o + V_{oy}t - \tfrac{1}{2}gt^2$$

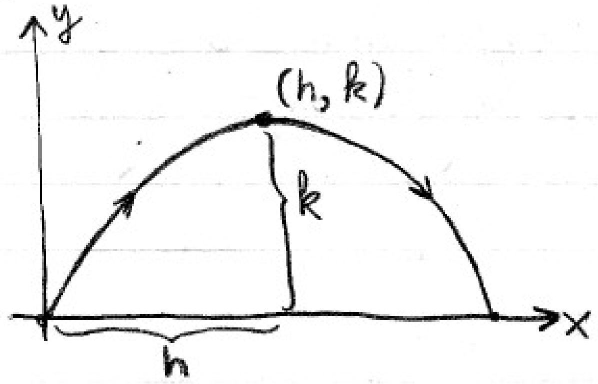
now technically the picture below is not the phase plane in the mathematical sense because we began with a system of <u>second</u> order ODEs. It is intersting we find parabolic motion in the xy-plane or the yv-plane (see $\boxed{E8}$). Let me do the algebra first,

$$t = \frac{x-x_o}{V_{ox}} \Rightarrow y - y_o = V_{oy}\left(\frac{x-x_o}{V_{ox}}\right) - \frac{1}{2}g\left(\frac{x-x_o}{V_{ox}}\right)^2$$

If it starts at the origin, $x_o = 0 = y_o$ this yields the familar eq^{n},

$$y = \frac{V_{oy}}{V_{ox}}x - \frac{1}{2}\frac{g}{V_{ox}^2}x^2 = \boxed{-\frac{1}{2}\frac{g}{V_{ox}^2}\left(x - \frac{V_{ox}V_{oy}}{g}\right)^2 + \frac{V_{oy}^2}{2g} = y}$$

Comparing the eq^{n} above to $y = A(x-h)^2 + k$ reveals



$$h = \frac{V_{ox}V_{oy}}{g}$$
$$k = V_{oy}^2/2g$$

STABILITY OF LINEAR SYSTEMS (§12.2)

A linear autonomous system has the form $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ in terms of matrix notation (§9.1-9.2) or explicitly,

$$\boxed{\begin{aligned} x'(t) &= a_{11}x + a_{12}y + b_1 \\ y'(t) &= a_{21}x + a_{22}y + b_2 \end{aligned}}$$

if the coefficient matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is nonsingular, $\det(A) \neq 0$ meaning $\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$ then we can argue there is only <u>one</u> critical point and we can even shift coordinates so the critical point is at the origin. Let's prove it (I'll assume you know some matrix techniques we'll cover soon, see §9.1 & 9.2 until then). Denote $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ so that $\vec{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$ then $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ as well and

$$\vec{x}' = A\vec{x} + \vec{b} \quad \text{for } \underline{\text{critical point}} \quad \vec{0} = A\vec{x} + \vec{b}$$
$$\Rightarrow A\vec{x} = -\vec{b}$$
$$\Rightarrow \vec{x} = -A^{-1}\vec{b}.$$

We know A^{-1} exists thanks to $\det(A) \neq 0$, in fact the formula for it is $A^{-1} = \frac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$. By assumption A & $\vec{b}$ are t-independent so

$$\frac{d}{dt}\left[\vec{x} + A^{-1}\vec{b}\right] = \frac{d\vec{x}}{dt} = \vec{x}'$$

So if we define $\vec{u} = \vec{x} + A^{-1}\vec{b}$ we find $\vec{x} = \vec{u} - A^{-1}\vec{b}$ and

$$\frac{d\vec{u}}{dt} = \frac{d\vec{x}}{dt} = A\vec{x} + \vec{b} = A(\vec{u} - A^{-1}\vec{b}) + \vec{b} = A\vec{u} - \cancel{AA^{-1}\vec{b}} + \cancel{\vec{b}}$$

Thus in the $\vec{u}$ coordinates $\frac{d\vec{u}}{dt} = A\vec{u}$ and the critical point is at $\vec{u} = 0$. Therefore, we can assume our general linear system has $\vec{b} = 0$ without loss of generality. We simply shift coordinates. (see Example 4 of pg. 752-753 for explicit details of how to shift coordinates)

As we just discussed the problem of isolated critical points of linear systems boils down to considering possible sol^{n}'s to

$$\boxed{\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}} \quad (*)$$

where $ad - bc \neq 0$ to insure $(0,0)$ is the only critical point.
It turns out $(*)$ can be written as $\vec{x}' = Ax$ and sol^{n}'s have the form $\vec{x} = e^{\lambda t}\vec{u}$ (usually, there is one exception) where λ and $\vec{u}$ may be complex. For λ & $\vec{u}$ to form the sol^{n} we'll learn they must satisfy (assuming $\vec{u} \neq 0$)

$$\boxed{(A - \lambda I)\vec{u} = 0} \quad \text{where} \quad \boxed{\det(A - \lambda I) = 0}$$

"eigenvector eq^{n}" "characteristic eq^{n}"

$\vec{u}$ - is eigenvector with eigenvalue - λ

The <u>characteristic eq^{n}</u> for this system with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ works out to,

$$\boxed{\lambda^2 - (a+d)\lambda + (ad - bc) = 0} \quad (**)$$

I just gave a short summary of what we do in chapter 9, for now I put off the details and instead focus on the graphical consequences of the various types of sol^{n}'s. These are summarized in the Th^{m}

Th^{m}/ Given a system such as $(*)$ with char. eq^{n} $(**)$ with sol^{n}'s λ_1, λ_2 the classification of the origin as a critical point goes as follows,

ROOTS ($\lambda =$)	TYPE OF CRITICAL POINT	STABILITY
distinct, positive	improper node	unstable
distinct, negative	improper node	asymptotically stable
opposite signs	saddle point	unstable
equal, positive	proper or improper node	unstable
equal, negative	proper or improper node	asymptotically stable
$\lambda = \alpha + i\beta$, $\alpha > 0$	spiral point	unstable
$\lambda = \alpha + i\beta$, $\alpha < 0$	spiral point	asymptotically stable
$\lambda = \alpha + i\beta$, $\alpha = 0$	center	stable

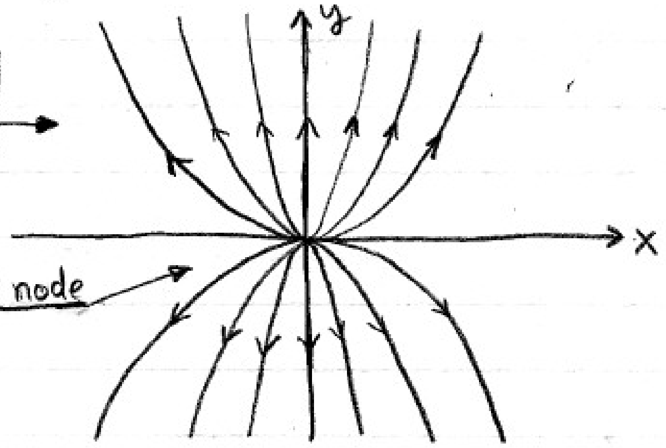
Rather than give a lengthy proof of the Th$^{\underline{m}}$ on PP10 we give examples of the various cases. I try to focus on rather simple cases, typically the more general pattern is a twist of the examples we consider (a twist which has much to do with the eigenvectors we'll play with soon).

$\boxed{E10}$

$$x' = 3x \Rightarrow x = c_1 e^{3t} \Rightarrow e^{3t} = \frac{x}{c_1}$$
$$y' = 6y \Rightarrow y = c_2 e^{6t} \Rightarrow y = c_2 (e^{3t})^2 = c_2 \left(\frac{x}{c_1}\right)^2$$

So in this case where $\lambda_1 = 3$, $\lambda_2 = 6$ (distinct & positive) the integral curves are $y = (c_2/c_1^2)\, x^2$

$\boxed{\lambda_1 = 3 \\ \lambda_2 = 6}$ →



improper node

(digression ↓)

Remark: the phase plane eq$^{\underline{n}}$ here is $\frac{dy}{dx} = \frac{6y}{3x}$ so then

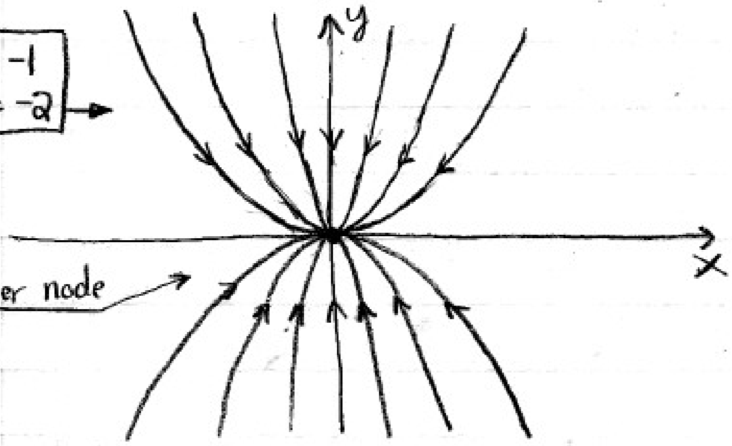
$$\int \frac{dy}{y} = \int \frac{2dx}{x}$$
$$\ln|y| = \ln x^2 + c$$
$$\therefore\ y = \pm e^{\ln x^2 + c} = k x^2$$

we can derive this w/o matrix methods if need be.

$\boxed{E11}$

$$x' = -x \Rightarrow x = c_1 e^{-t} \Rightarrow e^{-t} = x/c_1$$
$$y' = -2y \Rightarrow y = c_2 e^{-2t} = c_2 (x/c_1)^2 \ \therefore\ y = (c_2/c_1^2)\, x^2$$

$\boxed{\lambda_1 = -1 \\ \lambda_2 = -2}$ →



improper node

$x' = -x \Rightarrow$ that our sol$^{\underline{n}}$'s flow left for $x > 0$ and they flow right for $x < 0$. As you can see this makes <u>all</u> sol$^{\underline{n}}$'s flow towards the origin eventually.

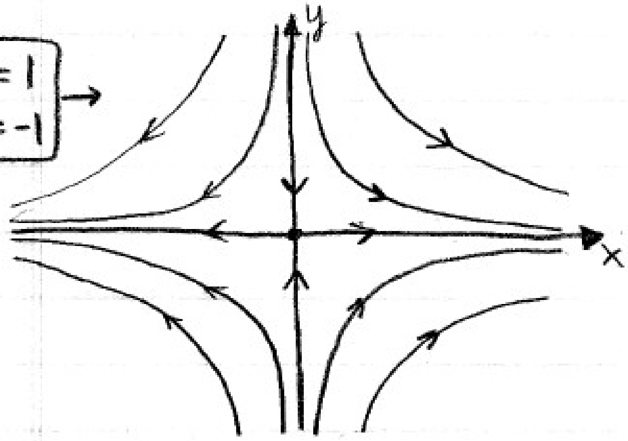
<u>origin is asymptotically stable</u>

$\boxed{E12}$ $x' = x \Rightarrow x = c_1 e^t \Rightarrow e^{-t} = c_1/x$ (for $x \neq 0$)

$y' = -y \Rightarrow y = c_2 e^{-t} = \frac{c_2}{c_1}\frac{1}{x}$ or $x = 0,\ y = c_2 e^{-t}$

or $y = 0,\ x = c_1 e^t$

$\boxed{\lambda_1 = 1 \\ \lambda_2 = -1}$ →

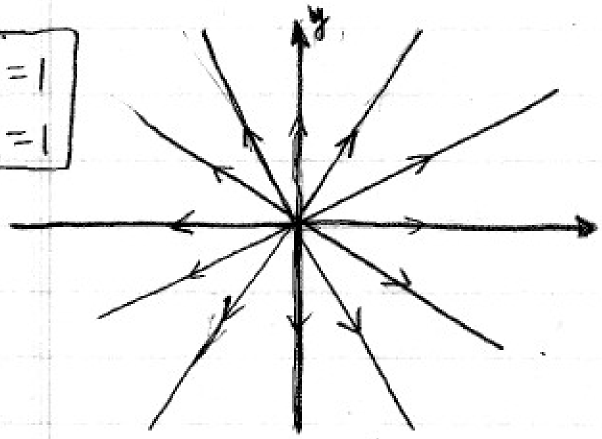


here sol^n's close to the origin may flow towards the origin or away. This makes (0,0) a saddle point. It is unstable because sol^n's flow towards & away from (0,0). (specifically I mean the sol^n's that lie along the x & y axes.

$\boxed{E13}$ $x' = x \Rightarrow x = c_1 e^t$

$y' = y \Rightarrow y = c_2 e^t \Rightarrow y = \frac{c_2}{c_1} x$

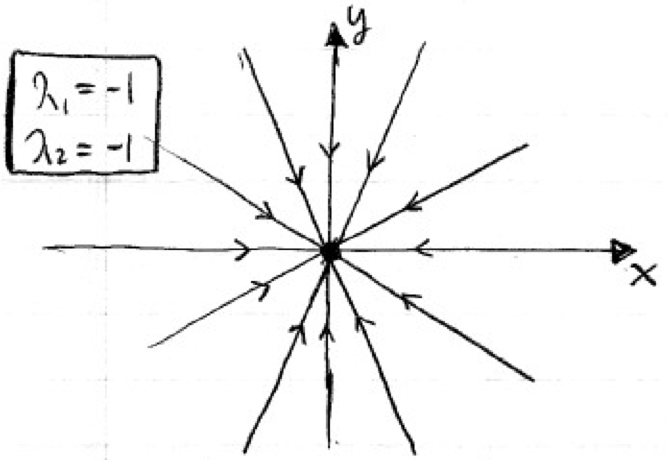
$\boxed{\lambda_1 = 1 \\ \lambda_2 = 1}$



all sol^n's flow away, this is an unstable point. It is a "proper" node since different sol^n's flowing through (0,0) have different slopes. As opposed to say $\boxed{E10}$ where we had an unstable "improper" node since most of the sol^n's flowing through (0,0) had the same slope (zero)

$\boxed{E14}$ $x' = -x \Rightarrow x = c_1 e^{-t}$

$y' = -y \Rightarrow y = c_2 e^{-t} \Rightarrow y = \left(\frac{c_2}{c_1}\right) x$



all sol^{n}'s eventually flow towards $(0,0)$ so this is an <u>asymptotically</u> <u>stable</u> <u>proper</u> <u>node</u>.

<u>Remark</u>: Th^{m} (1) says that when $\lambda_1 = \lambda_2 > 0$ or $\lambda_1 = \lambda_2 < 0$ we can get proper or improper nodes. The $\boxed{E13}$ & $\boxed{E14}$ show that if our sol^{n}'s have the form $x = c_1 e^{\lambda t}$, $y = c_2 e^{\lambda t}$ we'll get proper nodes ($\lambda > 0$ unstable, $\lambda < 0$ stable). The possibility of improper nodes follows from the unusual case that a t is present in the sol^{n} (we'll learn how the t comes from a generalized eigenvector later on). See page 748 and figure 12.9 for details on the graph in this case. Its improper since most sol^{n}'s have vertical tangent at $(0,0)$ in fig. 12.9.

Complex Case, when $\lambda = \alpha \pm i\beta$ where $\alpha, \beta \in \mathbb{R}$

I continue to follow §12.2 for the most part. Consider

$$\boxed{\begin{aligned} x' &= \alpha x - \beta y \\ y' &= \beta x + \alpha y \end{aligned}} \text{ — Eq}^{n}\text{(I)} \qquad \underline{\beta \neq 0 \text{ by assumption}}$$

In this case $A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ so $\det(A - \lambda I) = \det\begin{pmatrix} \alpha-\lambda & -\beta \\ \beta & \alpha-\lambda \end{pmatrix}$ giving characteristic eq^{n} $(\alpha-\lambda)^2 + \beta^2 = 0$ which is solved as follows $(\alpha-\lambda)^2 = -\beta^2 \Rightarrow \alpha - \lambda = \pm\sqrt{-\beta^2} = \pm i\beta \Rightarrow \lambda = \alpha \pm i\beta$.

Identify xy-plane with complex plane and write complex sol^{n}

$$z = x + iy \qquad (\text{identifying } z \text{ with } \vec{x} = [\begin{smallmatrix} x \\ y \end{smallmatrix}])$$

Observe then that if $\lambda = \alpha + i\beta$ then

$$\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt} = (\alpha x - \beta y) + i(\beta x + \alpha y) = (\alpha + i\beta)(x + iy) = \lambda z. \quad (*)$$

If we define ρ and θ by $z = \rho e^{i\theta}$ where ρ & θ are real fncts of t

$$\frac{dz}{dt} = \frac{\partial z}{\partial \rho}\frac{d\rho}{dt} + \frac{\partial z}{\partial \theta}\frac{d\theta}{dt} \quad : \text{chain rule from calc. III.}$$

$$= e^{i\theta}\frac{d\rho}{dt} + i\rho e^{i\theta}\frac{d\theta}{dt}$$

$$= \lambda z \quad : \text{using calculation } (*)$$

$$= (\alpha + i\beta)\rho e^{i\theta}$$

$$= \alpha\rho e^{i\theta} + i\beta\rho e^{i\theta}$$

Thus we find two real eq^{n}'s, cancelling $e^{i\theta}$'s everywhere,

$$\boxed{\frac{d\rho}{dt} = \alpha\rho \quad \& \quad \frac{d\theta}{dt} = \beta} \text{ — Eq}^{n}\text{ (I) in polar form}$$

We find sol^{n}'s by integration,

$$\boxed{\rho = c_1 e^{\alpha t} \quad \text{and} \quad \theta = \beta t + c_2}$$

When $\alpha \neq 0$ these give spirals, if $\alpha < 0$ then $\rho \to 0$ as $t \to \infty$, if $\alpha > 0$ then $\rho \to \infty$ as $t \to \infty$. In either case the spiral may go clockwise or counter-clockwise. The origin is called a spiral point in these cases. See figure 12.10 & 12.11 for picture.

Pure Imaginary Case

Continuing from PP15 we consider case $\alpha = 0$. This gives

$$\rho = c_1 \quad \text{and} \quad \frac{d\theta}{dt} = \beta$$

these are just circles about the origin. Let me prove this directly without the $z = x+iy$ trick. (which is nice btw) We seek to solve,

$$\boxed{E15} \qquad \frac{dx}{dt} = -\beta y \quad \& \quad \frac{dy}{dt} = \beta x$$

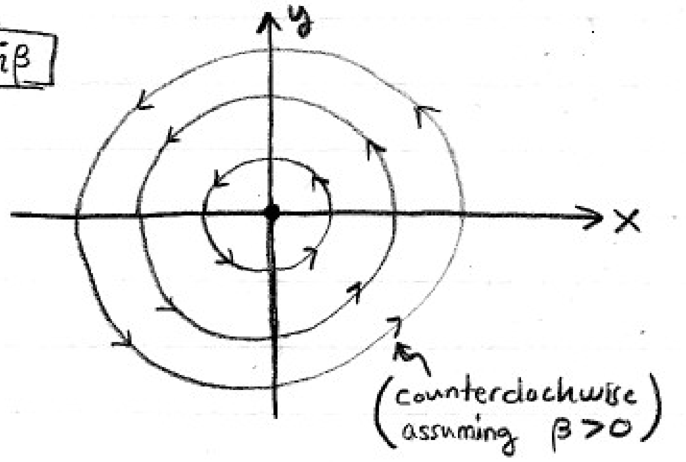
Differentiate to find $\frac{d^2x}{dt^2} = -\beta\frac{dy}{dt} = -\beta(\beta x)$. Well now we know what to do, $x'' + \beta^2 x = 0 \Rightarrow \lambda^2 + \beta^2 = 0 \Rightarrow \lambda = \pm i\beta$ thus $x = A\cos(\beta t + \phi)$. Now $\beta \neq 0$ by assumption so $1/\beta$ is well-defined,

$$y = \frac{-1}{\beta}\frac{dx}{dt} = \frac{-1}{\beta}\left[-\beta A \sin(\beta t + \phi)\right] = A\sin(\beta t + \phi) = y$$

It's not hard to see this is a circle with radius A,

$$x^2 + y^2 = A^2\cos^2(\beta t + \phi) + A^2 \sin^2(\beta t + \phi) = A^2$$

$\boxed{\lambda = i\beta}$



there is only one sol$^{\underline{n}}$ through $(0,0)$. It's the constant sol$^{\underline{n}}$

$$x = 0, \; y = 0$$

this is called a center it is stable

Remark: E10 → E15 have simple sol$^{\underline{n}}$s that didn't really involve matrix techniques. The interesting thing is that when we consider similar examples which have nontrivial coupling of x & y (i.e. you should use matrix techniques to solve) share the same graphs just twisted a bit.

ALMOST LINEAR SYSTEMS (§ 12.3)

Intuitively we have been speaking of stable and unstable critical points. Stable points draw sol^n's into them while unstable critical points have sol^n's that flow out of them. We can be more precise,

Def^n/ A critical point (x_0, y_0) of an autonomous system $[x' = f(x,y),\ y' = g(x,y)]$ is <u>stable</u> if, given any $\epsilon > 0$ there exists a $\delta > 0$ such that every sol^n $x = \phi(t)$ $y = \psi(t)$ of the system that satisfies

$$\sqrt{(\phi(0) - x_0)^2 + (\psi(0) - y_0)^2} < \delta$$

at $t = 0$ also satisfies

$$\sqrt{(\phi(t) - x_0)^2 + (\psi(t) - y_0)^2} < \epsilon$$

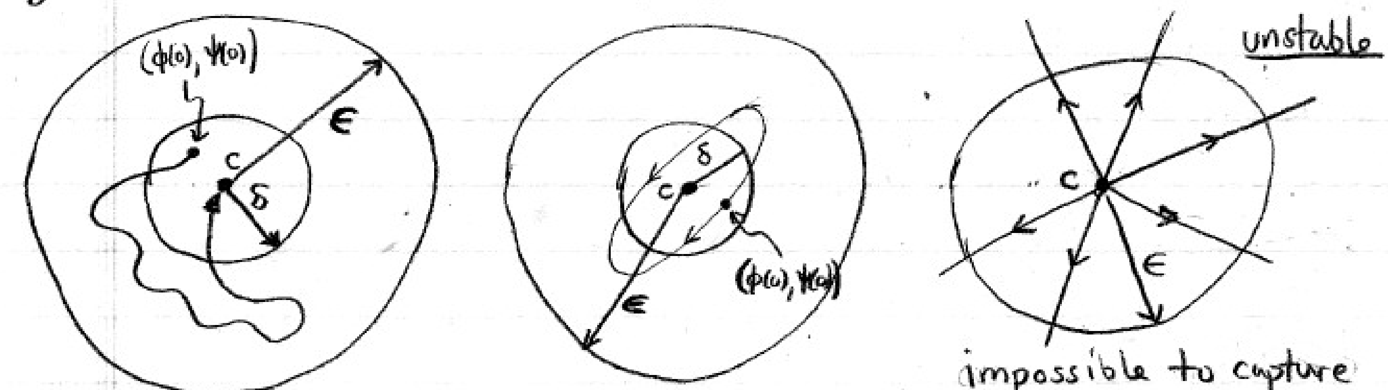
for $t \geq 0$. Furthermore, if (x_0, y_0) is stable and $\exists\, \eta > 0$ such that <u>every</u> sol^n $x = \phi(t)$, $y = \psi(t)$ that satisfies

$$\sqrt{(\phi(0) - x_0)^2 + (\psi(t) - y_0)^2} < \eta$$

at $t = 0$ also satisfies $\lim\limits_{t\to\infty} \phi(t) = x_0$ & $\lim\limits_{t\to\infty} \psi(t) = y_0$. then the critical point is asymptotically stable.

If a critical point is not stable then it is <u>unstable</u>.

Figure 12.14 in the text is helpful,



c is asymptotically stable

c is stable

impossible to capture sol^n's with ϵ-ball.

Given a nonlinear autonomous system we may be able to see linear behaviour close to some point, locally most systems become linear. To pin it down we define,

Def^n/ (ALMOST LINEAR SYSTEM) Let $(0,0)$ be a critical point of the autonomous system

$$\left.\begin{aligned} x'(t) &= ax + by + F(x,y) \\ y'(t) &= cx + dy + G(x,y) \end{aligned}\right\} \quad (*)$$

where a,b,c,d are constants and F, G are continuous in some disk about the origin. Assume $ad-bc \neq 0$ so $(0,0)$ is an isolated critical point. for the corresponding linear system ($x' = ax+by$, $y' = cx+dy$). Given all this set-up the system $(*)$ is <u>almost linear</u> near $(0,0)$ if

$$\frac{F(x,y)}{\sqrt{x^2+y^2}} \longrightarrow 0 \quad \text{and} \quad \frac{G(x,y)}{\sqrt{x^2+y^2}} \longrightarrow 0 \quad \text{as } \sqrt{x^2+y^2} \longrightarrow 0$$

$\boxed{E16}$ $x' = -x + x^2 + y^2$
$y' = -y + x^2 + y^2$

this is almost linear, $\boxed{E14}$ is the corresponding linear system. Here $F = x^2+y^2 = G$ notice $\frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$ this clearly goes to zero as $\sqrt{x^2+y^2} \longrightarrow 0$.

- table 12.1 on p. 759 tells us how $(0,0)$ behaves, it is an <u>improper</u> or <u>proper</u> or <u>spiral point</u> which is asymptotically stable. Which of these three choices? We can't know w/o more thought. These thoughts I leave to you & MAPLE or MATLAB.

$\boxed{\text{Th}^{\underline{m}} \text{ (By Poincaré) The stability properties for an almost linear system are the same as those for the linear system, with the exception of the pure imaginary case. In that case other techniques are needed to answer the question of stability}}$

- the possibilities are spelled out in TABLE 12.1, notice while the stability transfers to the almost linear case the Type of the critical point may change.

- Systems may be almost linear near points other than $(0,0)$. Essentially the idea is as follows,
  1.) find all critical points (for nonlinear systems can have more than one)
  2.) study each critical point seperately, shift coordinates as needed to make system almost linear about $(0,0)$ in new coordinates,
  3.) Study Eg^{b}/s and use Table 12.1 to guide in construction of global phase plane diagram.
  4.) be patient these can take a while, the example in the text takes up pgs. $759 \rightarrow 764$.

__Remark:__ I'm treating §12.3 rather lightly. I want you to know about Table 12.1 and to see if you can find agreement with the plots you create via software.