Normal Form

A system of ODEs is in <u>Normal Form</u> if it looks like

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\ &\vdots \\ x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t) \end{aligned}$$

Our using matrix notation with $A = (a_{ij}(t))$ the $n \times n$ <u>coefficients matrix</u>, $x(t) = (x_i(t))$ and $f(t) = (f_i(t))$,

$$x'(t) = A(t)x(t) + f(t)$$

<u>FACT</u>: Any <u>linear</u> system of ODEs can be written in normal form. This means by solving the normal form problem we are solving <u>all</u> linear systems of ODEs! The proof of this fact is best revealed by examples,

E1 Constants Coefficients 2nd order ODEs,

$$y'' + by' + cy = g(t)$$

We introduce new variables x_1, x_2 as follows,

$$x_1 = y$$
$$x_2 = y' \Rightarrow \underbrace{x_2 = x_1'}_{①}$$

Notice also

$$y'' + by' + cy = \underbrace{x_2' + bx_2 + cx_1 = g}_{②}$$

Rewrite ① & ②

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -cx_1 - bx_2 = g \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g \end{bmatrix}$$

A second order DEq has been rewritten as <u>two</u> first-order DEq's.

E2 $y^{(n)} + P_1 y^{(n-1)} + \cdots + P_n y = g$ n^{th} order ODE. Eq^n(1)

$x_1 = y$

$x_2 = y' \longrightarrow x_2 = x_1'$

$x_3 = y'' \longrightarrow x_3 = x_2'$

$x_4 = y''' \longrightarrow x_4 = x_3'$

$\vdots$

$x_n = y^{(n-1)} \longrightarrow x_n = x_{n-1}'$

Thus Eq^n(1) is the same as, $x' = Ax + f$,

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -P_n & -P_{n-1} & -P_{n-2} & \cdots & -P_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g \end{bmatrix}$$

oops, I used different numbering than text. No big deal, see pg. 524

E3 The same tricks work for systems of higher order linear differential eq^n's. For example,

$$2x'' + 6x - 2y = 0$$
$$y'' + 2y - 2x = 0$$

Let $x_1 = x$, $x_2 = x'$, $x_3 = y$, $x_4 = y'$. Notice that,

$x_1' = x_2$

$x_3' = x_4$

$2x_2' + 6x_1 - 2x_3 = 0 \rightarrow x_2' = -3x_1 + x_3$

$x_4' + 2x_3 - 2x_1 = 0 \rightarrow x_4' = 2x_1 - 2x_3$

Yielding

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Linear Operators, again

$$x' = Ax + f \iff x' - Ax = f$$

Introduce $L[x] = \frac{dx}{dt} - Ax$, then the DEq^n becomes,

$$\boxed{L[x] = f}$$

This looks like our operators before, but this operates on vector functions to spit-out new vector functions.

Def^n/ $L[x] = 0$ is homogeneous system of linear ODEs
$L[x] = f$ is nonhomogeneous system of linear ODEs

L is indeed a linear operator,

$$L[ax + by] = aL[x] + bL[y]$$

This means that if vector functions $x_1, x_2, \ldots, x_n$ are sol^n's to $L[x_i] = 0$ then

$$L[c_1 x_1 + c_2 x_2 + \cdots + c_n x_n] = \vec{0}$$

$$\vec{0} = c_1 L[x_1] + c_2 L[x_2] + \cdots + c_n L[x_n] = 0$$

the sum of homogeneous sol^n is a homogeneous sol^n.

Def^n/ The vector functions $x_1, x_2, \ldots, x_m$ are linearly dependent on an interval I if $\exists$ constants $c_1, c_2, \ldots, c_m$ not all zero such that

$$c_1 x_1(t) + c_2 x_2(t) + \cdots + c_m x_m(t) = 0 \quad \forall t \in I.$$

Otherwise they are linearly independent (LI)

Remark: vector functions $x_1, x_2, \ldots, x_m$ I also call vector-valued functions because their output is a vector. Each vector valued function has a bunch of plain old functions from $\mathbb{R} \to \mathbb{R}$. Assuming n-dim'l vector output,

$$x_j(t) = [x_{1j}(t), x_{2j}(t), \ldots, x_{nj}(t)]^T$$

^{T} makes this row a column.

Th^m (2) Given continous $A(t)$ and $f(t)$ on interval I containing point t_0. Then for any intial condition vector $X(t_0) = X_0$ $\exists$ a unique sol^n $X(t)$ on the whole interval I to the DEq^n $X' = AX + f$.

not very surprising given the correspondance we've seem to n^{th} order ODEs. But, this works for any $A(t)$ not just those $\simeq$ to n^{th} order single Deq^n

Def^n/ The Wronskian of n vector-valued functions $X_1, X_2, \dots, X_n$ is the real-valued function

$$W[X_1, X_2, \dots, X_n](t) = \begin{vmatrix} x_{1,1}(t) & x_{1,2}(t) & \cdots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & & x_{2,n}(t) \\ \vdots & & & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & & x_{n,n}(t) \end{vmatrix} = \det\left[X_1 | \cdots | X_n\right]$$

FACT: The Wronskian of sol^n's $X_1, X_2, \dots, X_n$ to $X' = AX$ on I is either always zero or never zero on I. Moreover, $X_1, X_2, \dots, X_n$ are a set of n-LI sol^n's to $X' = AX \iff W[X_1, \dots, X_n](t) \neq 0 \ \forall t \in I$.

E4 $\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{aligned} x &= c_1 e^t \\ y &= c_2 e^{2t} \\ z &= c_3 e^{3t} \end{aligned}$ $\left(\text{since } \begin{aligned} x' &= x \\ y' &= 2y \\ z' &= 3z \end{aligned}\right)$

We find then (using our crude reasoning on this easy problem) three vector sol^n's,

$$X_1 = \begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 \\ e^{2t} \\ 0 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 0 \\ 0 \\ e^{3t} \end{pmatrix}$$

Check LI with $W[X_1, X_2, X_3]$

$$W[X_1, X_2, X_3](t) = \begin{vmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{vmatrix} = e^t e^{2t} e^{3t} = e^{6t} \neq 0.$$

Since this is a homogeneous DEq^n we find sol^n

$$X_h = c_1 X_1 + c_2 X_2 + c_3 X_3 = \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \\ c_3 e^{3t} \end{pmatrix}.$$

Recall that for an n^{th} order linear DEq we found exactly n - LI sol^n's $Y_1, Y_2, \ldots, Y_n$. Something very similar occurs here.

Th^m(3) Given $X_1, X_2, \ldots, X_n$ LI sol^n's on I to the homogeneous system $X'(t) = A(t)X$, where $A(t)$ is continous on I. Then every sol^n to $X' = AX$ has the form

$$\boxed{X(t) = C_1X_1(t) + C_2X_2(t) + \cdots + C_nX_n(t)} \quad -(*)$$

where $C_1, C_2, \ldots, C_n$ are all constants

As before, we refer to $\{X_1, X_2, \ldots, X_n\}$ (as above) as the fundamental sol^n set for $X' = AX$. Notice we can restate the Th^m(3) in matrix language

$$\mathbf{X}(t) = \left[\, X_1(t) \,\middle|\, X_2(t) \,\middle|\, \cdots \,\middle|\, X_n(t) \,\right] \quad \text{"fundamental matrix"}$$

$$\mathbf{C} = [C_1, C_2, \ldots, C_n]^T \quad \left(\text{vector of arbitrary constants}\right)$$

Then $(*) \iff \boxed{X = \mathbf{X}\mathbf{C}}$ (a nice way of writing things.)

Th^m(4) Consider $\boxed{X'(t) = A(t)X(t) + f(t)}$ $(**)$ on interval I where $A(t)$ and $f(t)$ are continous. If $\{X_1, \ldots, X_n\}$ is the fundamental sol^n set to the corresponding homogeneous DEq^n $X'(t) = A(t)X(t)$ then the general sol^n to $(**)$

$$X(t) = X_p(t) + C_1X_1(t) + \cdots + C_nX_n(t)$$

Where $C_1, C_2, \ldots, C_n$ are constants and X_p is a particular sol^n to $(**)$. Suppressing t we say $X = X_p + \mathbf{X}\mathbf{C}$.

E5 Consider $\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$

We already found the corresponding homogeneous sol^n's in E4, recall that $\sum c = [c_1 e^t, c_2 e^{2t}, c_3 e^{3t}]^T$

I claim $x_p = [1 \ 1 \ 1]^T$ solves our eq^n here lets see,

$$x_p' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0$$

So it works. Thus by Th^m (4) the general sol^n is

$$x_g = x_p + \sum c = \begin{bmatrix} 1 + c_1 e^t \\ 2 + c_2 e^{2t} \\ 3 + c_3 e^{3t} \end{bmatrix}$$

Remark: to solve normal systems we have two tasks we need to address

1.) how to find $\sum(t)$ for $x' = Ax$

2.) how to find $x_p(t)$ with $x_p' = Ax_p + f$

After completing 1) & 2) we then assemble

3.) $x = x_p + \sum c$

And if we have intial data $x(t_0) = x_0$ then we'll have a fourth task as usual

4.) find c such that $x(t_0) = x_0$.

All of this parallels the previous ideas in this course. We will work on 1.) to begin, we'll focus on the case A is a constant matrix.