MA 341 – TEST III

- Show your work, write work on seperate paper.

PROBLEM ONE (15pts.) Consider the following second order ODE, $a, b, c \in \mathbb{R}$

$$aY'' + bY' + cY = 0$$

a.) convert this to a matrix differential eq$^{\underline{n}}$ in <u>normal form</u>

b.) show the auxillary eq$^{\underline{n}}$ = characteristic eq$^{\underline{n}}$ where the aux. eq$^{\underline{n}}$ is the eq$^{\underline{n}}$ we studied earlier in the course whereas the char. eq$^{\underline{n}}$ comes from the matrix formalism.

PROBLEM TWO (15pts.) Find the general sol$^{\underline{n}}$ to the following system of differential eq$^{\underline{n}}$'s.

$$X'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} X(t) + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

PROBLEM THREE (15pts.) Find the general sol$^{\underline{n}}$ to the following system of differential eq$^{\underline{n}}$'s

$$X'(t) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix} X(t)$$

PROBLEM FOUR (15pts.) Compute the matrix exponential e^{At} for the following matrix then write the general sol$^{\underline{n}}$ to $x' = Ax$

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

PROBLEM FIVE (15pts.) These questions are more conceptual, complete the others first.

(a.) <u>Show</u> that if $Au = 3u$ and $Av = 2v$ then u and v are linearly independent.

(b.) Is it possible that the matrix below is a fundamental matrix, <u>explain why/why not.</u>

$$\mathbb{X} = \begin{bmatrix} e^t & e^t \\ e^{3t} & e^{3t} \end{bmatrix}$$

(c.) Given $(A-3I)^3 u_3 = (A-3I)^2 u_2 = (A-3I) u_1 = 0$ <u>calculate three LI sol$^{\underline{n}}$'s to $x' = Ax$.</u>

Ma 341 – Test III – Sol$^{\underline{n}}$

PROBLEM ONE Convert $aY'' + bY' + cY = 0$ into a matrix DEq$^{\underline{n}}$ in normal form.

a.) $X_1 = Y$

$X_2 = Y' = X_1'$

$X_2' = Y'' = -\frac{b}{a}Y' - \frac{c}{a}Y = -\frac{b}{a}X_2 - \frac{c}{a}X_1$

$$\underbrace{\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}'}_{X'} = \underbrace{\begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}_{X} \longleftrightarrow X' = AX$$

b.)

$$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -c/a & -b/a - \lambda \end{bmatrix}$$

$$= -\lambda\left(-\frac{b}{a} - \lambda\right) + \frac{c}{a} = 0 \Longleftrightarrow \boxed{a\lambda^2 + b\lambda + c = 0}$$

assuming $a \neq 0$. we just had to multiply by a to see that the auxillary and characterstic eq$^{\underline{n}}$s are the same.

PROBLEM TWO Solve $X' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} X + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

To begin we find the fundamental sol$^{\underline{n}}$ set for $X' = AX$,

$$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0 \quad \therefore \lambda = \pm i$$

<u>Find e-vector $u_1 = a + ib$ for $\lambda = i$</u>

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} -iu - v = 0 \\ u - iv = 0 \end{matrix} \longrightarrow v = -iu$$

Choose $u = 1 \Rightarrow u_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$$\Rightarrow u_1 = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{a} + i\underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{b}$$

but $iu_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is another answer

PROBLEM TWO Continued

Assemble the sol^{n}s using the a & b just found

$$\vec{X}_1 = \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$\vec{X}_2 = \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos t \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$$

This gives us the fundamental matrix $\mathbf{X}$ for $x' = Ax$,

$$\mathbf{X} = \left[\vec{X}_1 \,\middle|\, \vec{X}_2 \right] = \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix}$$

$$\mathbf{X}^{-1} = \frac{1}{-\cos^2 t - \sin^2 t} \begin{bmatrix} -\cos t & -\sin t \\ -\sin t & \cos t \end{bmatrix} = \underline{\begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix} = \mathbf{X}^{-1}(t)}$$

we'll need this shortly.

Variation of parameters for systems says that

$$x' = Ax + f \quad \text{has sol}^{n} \quad x_p = \mathbf{X} \int \mathbf{X}^{-1} f \, dt$$

Applying that to our problem here

$$x_p = \mathbf{X} \int \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} dt$$

$$= \mathbf{X} \int \begin{bmatrix} 2\cos t + 3\sin t \\ 2\sin t - 3\cos t \end{bmatrix} dt$$

$$= \mathbf{X} \begin{bmatrix} 2\sin t - 3\cos t \\ -2\cos t - 3\sin t \end{bmatrix}$$

$$= \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix} \begin{bmatrix} 2\sin t - 3\cos t \\ -2\cos t - 3\sin t \end{bmatrix} \qquad (2\times 2)(2\times 1) = (2\times 1)$$

$$=$$

$$= \begin{bmatrix} \cos t(2\sin t - 3\cos t) + \sin t(-2\cos t - 3\sin t) \\ \sin t(2\sin t - 3\cos t) - \cos t(-2\cos t - 3\sin t) \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{2\sin t\cos t} - 3\cos^2 t - \cancel{2\sin t\cos t} - 3\sin^2 t \\ 2\sin^2 t - \cancel{3\sin t\cos t} + 2\cos^2 t + \cancel{3\cos t\sin t} \end{bmatrix}$$

$$\underline{x_p = \begin{bmatrix} -3 \\ 2 \end{bmatrix}}$$

PROBLEM TWO CONCLUSION

We just found $X_p = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Thus

$$\boxed{X = C_1 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + C_2 \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix}}$$

PROBLEM THREE Find general sol^{n} to $X' = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} X$

$$\det[A - \lambda I] = \det \begin{bmatrix} -1-\lambda & 0 & 0 \\ 2 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix}$$

$$= -(\lambda+1)\left((2-\lambda)^2 - 1\right)$$

$$= -(\lambda+1)(\lambda^2 - 4\lambda + 4 - 1)$$

$$= -(\lambda+1)(\lambda-3)(\lambda-1) \quad \therefore \quad \underline{\lambda_1 = -1, \ \lambda_2 = 3, \ \lambda_3 = 1}$$

Find $u_1 = (u, v, w)^T$ with $(A + I)u_1 = 0$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 2u + 3v + w = 0 &\rightarrow w = -2u + -3v \\ v + 3w = 0 &\rightarrow 9w = -3v \end{aligned}\right\} \rightarrow 8w = 2u$$

After the algebra above we see putting everything in terms of w should make life easy

$$\begin{aligned} u &= 4w \\ v &= -3w \\ w &= w \end{aligned} \xrightarrow[w=1]{\text{choose}} \boxed{u_1 = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}}$$

PROBLEM THREE

Still have two more e-vectors to find,

Find $u_2 = (u, v, w)^T$ with $(A - 3I)u_2 = 0$

$$\begin{bmatrix} -4 & 0 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-4u = 0 \longrightarrow u = 0$

$2u - v + w = 0 \longrightarrow v = w$

choose $v = 1 = w$ $\Rightarrow$ $\boxed{u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$ $(\lambda_2 = 3)$

Find $u_3 = (u, v, w)^T$ with $(A - I)u_3 = 0$

$$\begin{bmatrix} -2 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2u = 0 \longrightarrow u = 0$

$2u + v + w = 0 \longrightarrow v = -w$

choose $v = 1 \Rightarrow w = -1$ $\Rightarrow$ $\boxed{u_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}$ $(\lambda_3 = 1)$

Thus the general $\text{sol}^{\underline{n}}$ is

$$\boxed{x = c_1 e^{-t}\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}$$

PROBLEM FOUR Compute e^{At} for $x' = Ax$ and write the gen sol^n

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

Lets find e-values / e-vectors and work from there.

$$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix} = -\lambda(1-\lambda) = 0$$

$$\therefore \underline{\lambda_1 = 0, \ \lambda_2 = 1}$$

Find $u_1 = (u, v)^T$ with $Au_1 = 0$

$$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} 2v = 0 \\ v = 0 \end{matrix} \rightarrow u \text{ is free} \rightarrow \boxed{u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

Find $u_2 = (u, v)^T$ with $(A - I)u_2 = 0$

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} -u + 2v = 0 \\ \therefore u = 2v \end{matrix} \rightarrow \boxed{u_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

Then we can already write the general sol^n even though we've not computed e^{At} yet.

$$\boxed{x = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

Thus $\Phi = \begin{bmatrix} 1 & 2e^t \\ 0 & e^t \end{bmatrix}$ and $\Phi^{-1} = \frac{1}{e^t}\begin{bmatrix} e^t & -2e^t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & e^{-t} \end{bmatrix}$

Now recall that $e^{At} = \Phi C$ then we could solve for C by looking at $t = 0$ where $\exp(0 \cdot t) = I = \Phi(0)C$ thus $C = \Phi^{-1}(0)$ and $\underline{e^{At} = \Phi(t)\Phi^{-1}(0)}$. Use this to compute e^{At} (we had hwk like this)

$$e^{At} = \begin{bmatrix} 1 & 2e^t \\ 0 & e^t \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & e^{-0} \end{bmatrix} = \boxed{\begin{bmatrix} 1 & -2 + 2e^t \\ 0 & e^t \end{bmatrix} = e^{At}}$$

PROBLEM FIVE (For problem (a.) I should have added that $u, v \neq 0$.)

a.) Suppose A, u, v are nontrivial and $Au = 3u$ & $Av = 2v$

Suppose $\exists$ a constant c which has $u = cv$

$$\begin{array}{ccccc} Au & = & Acv & = & cAv \\ \| & & & & \| \\ 3u & & & & c2v = 2cv = 2u \end{array}$$

Therefore $3u = 2u \Rightarrow u = 0$ which is impossible since u is an e-vector. $\therefore$ Its not possible to write $u = cv$

$\therefore$ u, v are LI.

(b.) $X = \begin{bmatrix} e^t & e^t \\ e^{3t} & e^{3t} \end{bmatrix}$ has $\det(X) = e^{4t} - e^{4t} = 0$

$\therefore$ X^{-1} does not exist

$\therefore$ X is <u>not</u> a fundamental matrix.

(c.) Given $(A-3I)u_3 = u_2$, $(A-3I)u_2 = u_1$ and $(A-3I)u_1 = 0$ (as I said in class) then it follows that

$$(A-3I)^3 u_3 = (A-3I)^2 u_2 = (A-3I)u_1 = 0$$

Then we know u_1, u_2, u_3 are LI assuming that $u_1, u_2, u_3 \neq 0$ as we always do,

$$X_1 = e^{At} u_1 = e^{3t}\left(u_1 + t(A-3I)u_1 + \cdots\right) = \boxed{e^{3t} u_1 = X_1}$$

$$X_2 = e^{At} u_2 = e^{3t}\left(u_2 + t(A-3I)u_2\right) = \boxed{e^{3t}(u_2 + t u_1) = X_2}$$

$$X_3 = e^{At} u_3 = e^{3t}\left(u_3 + t(A-3I)u_3 + \tfrac{1}{2}t^2(A-3I)^2 u_3\right)$$

$$= e^{3t}\left(u_3 + t u_2 + \tfrac{1}{2}t^2 (A-3I) u_2\right)$$

$$= \boxed{e^{3t}\left(u_3 + t u_2 + \tfrac{1}{2}t^2 u_1\right) = X_3}$$

We've used the usual trick,

$$e^{At} = e^{At - \lambda I t + \lambda I t} = e^{\lambda I t} e^{(A-\lambda I)t} = e^{\lambda t}\left(I + t(A-\lambda I) + \tfrac{1}{2}t^2(A-\lambda I)^2 + \cdots\right)$$