YOUR NAME HERE:

MA 341-004, Introduction to Differential Equations

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Test III: Systems of ODEs
Date: Wednesday, June 20, 2007

Directions: Show your work, if you doubt that you've shown enough detail then ask.

1. ~~(25pts)~~ (30pts) Find the eigenvalues and eigenvectors of the matrix below

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$$

then find the general solution to $\frac{d\vec{x}}{dt} = A\vec{x}$.

2. (10pts) Suppose that $y'' + 2y' + 5y = t$. Restate this second order differential equation as a system of two first order differential equations. Then write the system as a matrix differential equation. For 1pt show that the auxillary and characteristic equations are the same for this example.

3. (10 pts) Show that $\vec{x_p} = X\vec{v}$ is a solution to $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$ if,

$$\vec{x_p}(t) = X(t) \int X^{-1}(t)\vec{f}(t)dt.$$

Here we assume X is a fundamental matrix for the system.

4. ~~(20pts)~~ (30pts) Solve $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$ given that

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \vec{f} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

5. (20pts) Suppose that A is a 5x5 matrix such that $det(A - \lambda I) = (\lambda-1)^2(\lambda-3)^3$. Furthermore, suppose that

$$(A - I)\vec{u_1} = 0 \qquad and \qquad (A - I)\vec{u_2} = 0$$

where $\vec{u_1}, \vec{u_2}$ are nontrivial and linearly independent. Next suppose that,

$$(A - 3I)\vec{u_3} = 0 \qquad and \qquad (A - 3I)\vec{u_4} = \vec{u_3} \qquad and \qquad (A - 3I)\vec{u_5} = \vec{u_4}$$

where $\vec{u_3}, \vec{u_4}, \vec{u_5}$ are all nontrivial. Given all this data calculate the general solution to $\frac{d\vec{x}}{dt} = A\vec{x}$ in terms of the given vectors. You may use the formula on the board without proof. However, you should certainly show your work.

6. (BONUS 5pts) Suppose that $\lambda = \alpha + i\beta$ ($\alpha, \beta \in \mathbb{R}$) is a complex eigenvalue of a matrix A. Furthermore suppose that we have two nontrivial complex vectors $\vec{u_1}$ and $\vec{u_2}$ such that

$$(A - \lambda I)\vec{u_1} = 0 \qquad (A - \lambda I)\vec{u_2} = \vec{u_1}.$$

Derive the four *real* solutions that follow from this data. Use the notation $\vec{u_1} = \vec{a_1} + i\vec{b_1}$ and $\vec{u_2} = \vec{a_2} + i\vec{b_2}$ where $\vec{a_i}, \vec{b_j}$ are real vectors.

TEST III SOLUTION : SYSTEMS OF ODEs

PROBLEM ONE To begin find eigenvalues of $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 1 \\ 0 & 2 & 4-\lambda \end{vmatrix} = (1-\lambda)\left[(3-\lambda)(4-\lambda) - 2\right]$$

$$= (1-\lambda)\left[10 - 7\lambda + \lambda^2\right]$$

$$= (1-\lambda)(\lambda-5)(\lambda-2) = 0$$

$$\boxed{\lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 5}$$

Eigenvector $\vec{u}_1 = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ with $A\vec{u}_1 = \vec{u}_1$ has $(A - I)\vec{u}_1 = 0$ so,

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow - \begin{pmatrix} 2u + 2v + w = 0 \\ 2v + 3w = 0 \end{pmatrix}$$

$$2u - 2w = 0 \quad \therefore \quad \underline{u = w}$$

$$\Rightarrow v = -\tfrac{3}{2}w = -\tfrac{3}{2}u.$$

$$\vec{u}_1 = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u \\ -3/2\, u \\ u \end{bmatrix} = u\begin{bmatrix} 1 \\ -3/2 \\ 1 \end{bmatrix}$$

choose $\boxed{\vec{u}_1 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}}$

Eigenvector $\vec{u}_2 = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ satisfies $(A - 2I)\vec{u}_2 = 0$

$$\begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -u = 0 \\ 2u + v + w = 0 \\ 2v + 2w = 0 \end{matrix} \rightarrow v = -w$$

$$\therefore \vec{u}_2 = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ -w \\ w \end{bmatrix}$$

choose $\boxed{\vec{u}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$

Eigenvector $\vec{u}_3 = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ satisfies $(A - 5I)\vec{u}_3 = 0$

$$\begin{bmatrix} -4 & 0 & 0 \\ 2 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -4u = 0 \\ 2u - 2v + w = 0 \\ 2v - w = 0 \end{matrix} \Rightarrow w = 2v$$

$$\therefore \vec{u}_3 = \begin{bmatrix} 0 \\ v \\ 2v \end{bmatrix} \Rightarrow \boxed{\vec{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}$$

Thus we find $\vec{x}' = Ax$ has the sol$^{\underline{n}}$,

$$\boxed{\vec{x} = c_1 e^{t}\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{5t}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}$$

PROBLEM TWO

Convert $y'' + 2y' + 5y = t$ to a matrix DEq^{n},

$$x_1 = y$$
$$x_2 = y' = x_1'$$
$$x_2' = y'' = -2y' - 5y + t = -2x_2 - 5x_1 + t$$

$$x_1' = x_2$$
$$x_2' = -5x_1 - 2x_2 + t$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ t \end{bmatrix}$$

The aux. eq^{n} is $\lambda^2 + 2\lambda + 5 = 0$. Is the characteristic eq^{n} the same?

$$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -5 & -2-\lambda \end{bmatrix}$$
$$= -\lambda(-2-\lambda) + 5$$
$$= \underline{\lambda^2 + 2\lambda + 5 = 0}. //$$

PROBLEM THREE

Let $\vec{X}_p = \mathbb{X}\vec{v}$ is a sol$^{\underline{n}}$ to $x' = A\vec{x} + \vec{f}$ provided that $\mathbb{X}$ is a fundamental matrix and

$$\vec{v}(t) = \mathbb{X}^{-1}(t) \int \mathbb{X}(t)\vec{f}(t)\,dt$$

Consider then

$$\frac{dx_p}{dt} = \frac{d}{dt}\left[\mathbb{X}\vec{v}\right]$$

$$= \frac{d\mathbb{X}}{dt}\vec{v} + \mathbb{X}\frac{d\vec{v}}{dt}$$

$$= \cancel{A\mathbb{X}\vec{v}} + \mathbb{X}\frac{d\vec{v}}{dt} = \cancel{A\mathbb{X}\vec{v}} + \vec{f}$$

$$\mathbb{X}\frac{d\vec{v}}{dt} = \vec{f}$$

note $\mathbb{X}^{-1}$ exists since $\mathbb{X}$ is a fundamental matrix which is nonsingular since it is made of n- LI sol$^{\underline{n}}$s. (By def$^{\underline{n}}$)

$$\Rightarrow \mathbb{X}^{-1}\mathbb{X}\frac{d\vec{v}}{dt} = \mathbb{X}^{-1}\vec{f} \quad \& \quad \mathbb{X}^{-1}\mathbb{X} = I.$$

$$\Rightarrow \vec{v} = \int \frac{d\vec{v}}{dt}\,dt = \int \mathbb{X}^{-1}\vec{f}\,dt. \quad //$$

$\boxed{\text{PROBLEM FOUR}}$ Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\vec{f} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ solve the nonhomogeneous system $\vec{x}' = A\vec{x} + \vec{f}$. We can use formula in $\boxed{\text{PROBLEM 3}}$, need to find Φ.

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \quad \therefore \lambda = \pm i \quad \text{use } \underline{\lambda = i}$$

$\alpha = 0,\ \beta = 1$

$$(A - iI)\vec{u} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-iu + v = 0 \quad \therefore \quad v = iu \Rightarrow \vec{u} = \begin{bmatrix} u \\ iu \end{bmatrix}$$

can choose $\vec{u} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{a}} + i \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{b}}$.

Thus

$$\vec{x}_1 = \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

$$\vec{x}_2 = \cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

Hence $\Phi = [\vec{x}_1 | \vec{x}_2] = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$ note $\det(\Phi) = \cos^2 t + \sin^2 t = 1$.

$\therefore \quad \Phi^{-1} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$: using f-la on board.

Now calculate,

$$\vec{v} = \Phi \int \Phi^{-1} \vec{f}\, dt = \Phi \int \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} dt$$

$$= \Phi \int \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} dt$$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\sin^2 t - \cos^2 t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

$\boxed{\text{PROBLEM FOUR CONTINUED}}$

$$\begin{aligned}\vec{x}(t) &= \Phi c + \Phi v \\ &= c_1\vec{X}_1 + c_2\vec{X}_2 + \begin{bmatrix}\cos t & \sin t \\ -\sin t & \cos t\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} \\ &= \boxed{c_1\begin{bmatrix}\cos t \\ -\sin t\end{bmatrix} + c_2\begin{bmatrix}\sin t \\ \cos t\end{bmatrix} + \begin{bmatrix}\sin t \\ \cos t\end{bmatrix} = \vec{x}(t)}\end{aligned}$$

$\boxed{\text{PROBLEM FIVE}}$ Given A is 5×5 and $\det(A-\lambda I) = (\lambda-1)^2(\lambda-3)^2$ this gives eigenvalues $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = \lambda_5 = 3$.

$$\left.\begin{aligned}(A-I)\vec{u}_1 = 0 &\Rightarrow \vec{x}_1 = e^t\vec{u}_1 \\ (A-I)\vec{u}_2 = 0 &\Rightarrow \vec{x}_2 = e^t\vec{u}_2\end{aligned}\right\}$$ follows from eq^{n} below actually

Next as given on the board

$$\begin{aligned}e^{At}\vec{u} &= e^{\lambda t}\left(I + t(A-\lambda I) + \tfrac{1}{2}t^2(A-\lambda I)^2 + \cdots\right)\vec{u} \\ &= e^{\lambda t}\vec{u} + t(A-\lambda I)\vec{u} + \tfrac{1}{2}t^2(A-\lambda I)^2\vec{u} + \cdots\end{aligned}$$

Now $(A-3I)\vec{u}_3 = 0$, $(A-3I)\vec{u}_4 = \vec{u}_3$ and $(A-3I)\vec{u}_5 = \vec{u}_4$

$$\vec{X}_3 = e^{At}\vec{u}_3 = e^{3t}\vec{u}_3 + \cancel{t(A-3I)\vec{u}_3}^{\,0} + \cdots = \boxed{e^{3t}\vec{u}_3 = \vec{X}_3}$$

$$\begin{aligned}\vec{X}_4 = e^{At}\vec{u}_4 &= e^{3t}\vec{u}_4 + t(A-3I)\vec{u}_4 + \tfrac{1}{2}t^2(A-3I)^2\vec{u}_4 + \cdots \\ &= e^{3t}\vec{u}_4 + t\vec{u}_3 + \tfrac{1}{2}t^2\cancel{(A-3I)\vec{u}_3}^{\,0} \\ &= \boxed{e^{3t}\vec{u}_4 + t\vec{u}_3 = \vec{X}_4}\end{aligned}$$

$$\begin{aligned}\vec{X}_5 = e^{At}\vec{u}_5 &= e^{3t}\vec{u}_5 + t(A-3I)\vec{u}_5 + \tfrac{1}{2}t^2(A-3I)^2\vec{u}_5 + \tfrac{1}{3!}t^3(A-3I)^3\vec{u}_5 \\ &= e^{3t}\vec{u}_5 + t\vec{u}_4 + \tfrac{1}{2}t^2(A-3I)\vec{u}_4 + \tfrac{1}{3!}t^3\cancel{(A-3I)^2\vec{u}_4}^{\,0} \\ &= \boxed{e^{3t}\vec{u}_5 + t\vec{u}_4 + \tfrac{1}{2}t^2\vec{u}_3 = \vec{X}_5}\end{aligned}$$

The general sol^{n} is $\vec{X} = c_1\vec{X}_1 + c_2\vec{X}_2 + c_3\vec{X}_3 + c_4\vec{X}_4 + c_5\vec{X}_5$ where $\vec{X}_i$ are as above.

Bonus: $\lambda = \alpha + i\beta$ and $\vec{u}_j = \vec{a}_j + i\vec{b}_j \quad j = 1,2$
plus we're given $(A - \lambda I)\vec{u}_2 = \vec{u}_1$ and $(A-\lambda I)\vec{u}_1 = 0$.

Complex Sol^{n}'s are easy to find,

$$\vec{W}_1 = e^{At}\,\vec{u}_1$$
$$\vec{W}_2 = e^{At}\,\vec{u}_2$$

Then within $\vec{W}_1$ & $\vec{W}_2$ are four real sol^{n}'s
namely $Re\{\vec{W}_1\}$, $Im\{\vec{W}_1\}$ ← we found these in lecture,

$$\vec{W}_1 = e^{(\alpha+i\beta)t}(\vec{a}_1 + i\vec{b}_1)$$
$$= \underbrace{e^{\alpha t}(\cos(\beta t)\vec{a} - \sin(\beta t)\vec{b})}_{\vec{X}_1 = Re\{\vec{W}_1\}} + i\underbrace{e^{\alpha t}(\sin(\beta t)\vec{a} + \cos(\beta t)\vec{b})}_{\vec{X}_2 = Im\{\vec{W}_1\}}$$

Now $\vec{W}_2$ is new, we didn't discuss it in lecture,

$$\vec{W}_2 = e^{At}\vec{u}_2$$
$$= e^{(\alpha+i\beta)t}\left(\vec{u}_2 + t(A - (\alpha+i\beta)I)\vec{u}_2 + \frac{t^2}{2}\underbrace{(A-\lambda I)^2\vec{u}_2}_{0} + \cdots\right)$$
$$= e^{(\alpha+i\beta)t}(\vec{u}_2 + t\vec{u}_1)$$
$$= e^{(\alpha+i\beta)t}([\vec{a}_2 + t\vec{a}_1] + i[\vec{b}_2 + t\vec{b}_1])$$

By our calculation for $\vec{W}_1$ in lecture we see

$$\vec{X}_3 = Re\{\vec{W}_2\} = e^{\alpha t}((\cos\beta t)(\vec{a}_2 + t\vec{a}_1) - (\sin\beta t)(\vec{b}_2 + t\vec{b}_1))$$

$$\vec{X}_4 = Re\{\vec{W}_2\} = e^{\alpha t}((\sin\beta t)[\vec{a}_2 + t\vec{a}_1] + [\cos\beta t][\vec{b}_2 + t\vec{b}_1])$$