Math 341
Exam 2

Name:________________________

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his exam contains 4 problems. Do all of your work on your own paper, please leave white space between problen
In problems 1–4 partial credit will be given. SHOW YOUR WORK.

Problem 1

(25 pts) Calculate the Laplace transforms of the following functions,

(a.) $f(t) = te^{-t} - 3$

(b.) $f(t) = 13e^{2t}sin(t + \pi)$

(c.) $f(t) = e^{-2t}tsin(3t)$

(d.) $f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ t^2, & 1 \le t \le 2 \\ sin(t), & 2 < t \end{cases}.$

(e.) $f(t) = sin(t)cos^2(t)$

Problem 2

(36 pts) Compute the inverse Laplace transforms of, (use a convolution to state your answer for (d.), do NOT calculate the integrals implicit within those convolutions)

(a.)

$$F(s) = \frac{s+1}{s^2 - 2s + 5}$$

(b.)

$$F(s) = \frac{7s^3 - 2s^2 - 3s + 6}{s^3(s-2)}$$

(c.)

$$F(s) = \frac{e^{-s}(4s+2)}{s(s+1)}$$

(d.)

$$F(s) = \frac{100s(s+2)}{(s^2+25)((s+2)^2+16)(s-1)}$$

Problem 3

(36 pts) Solve the initial value problems by the method of Laplace transforms.

(a.) $y'' + y = 2e^t$, $y(0) = 1$, $y'(0) = 2$

(b.) $y'' - 2y' + y = u(t-1)$, $y(0) = 0$ and $y'(0) = 1$ (u(t) is the unit-step function)

(c.) $w'' + w = \delta(t - \pi)$, $w(0) = 0$ and $w'(0) = 0$ (δ the Dirac Delta function)

Problem 4

(4 pts) Show explicitly from the defintion of the the Laplace transform that for $a \geq 0$,

$$\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}.$$

TABLE 7.1 BRIEF TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$
1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n, $n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}$, $s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}$, $s > 0$
$e^{at}t^n$, $n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$

TABLE 7.2 PROPERTIES OF LAPLACE TRANSFORMS

$\mathscr{L}\{f+g\} = \mathscr{L}\{f\} + \mathscr{L}\{g\}$.

$\mathscr{L}\{cf\} = c\mathscr{L}\{f\}$ for any constant c .

$\mathscr{L}\{e^{at}f(t)\}(s) = \mathscr{L}\{f\}(s-a)$.

$\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\}(s) - f(0)$.

$\mathscr{L}\{f''\}(s) = s^2\mathscr{L}\{f\}(s) - sf(0) - f'(0)$.

$\mathscr{L}\left\{f^{(n)}\right\}(s) = s^n\mathscr{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$.

$\mathscr{L}\{t^nf(t)\}(s) = (-1)^n\frac{d^n}{ds^n}\left(\mathscr{L}\{f\}(s)\right)$.

Thus, for $a \geq 0$,

(6) $\mathscr{L}\{\delta(t-a)\}(s) = e^{-as}$.

then gives

(8) $\mathscr{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathscr{L}\{g(t+a)\}(s)$.

The Laplace transform of $u(t-a)$ with $a \geq 0$ is

(4) $\mathscr{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s}$,

TRANSLATION IN t

Theorem 8. Let $F(s) = \mathscr{L}\{f\}(s)$ exist for $s > \alpha \geq 0$. If a is a positive constant, then

(5) $\mathscr{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)$,

and, conversely, an inverse Laplace transform† of $e^{-as}F(s)$ is given by

(6) $\mathscr{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a)$.

TRANSFORM OF PERIODIC FUNCTION

Theorem 9. If f has period T and is piecewise continuous on $[0, T]$, then

(12) $$\mathscr{L}\{f\}(s) = \frac{F_T(s)}{1-e^{-sT}} = \frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}} .$$

CONVOLUTION THEOREM

Theorem 11. Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order α and set $F(s) = \mathscr{L}\{f\}(s)$ and $G(s) = \mathscr{L}\{g\}(s)$. Then

(8) $\mathscr{L}\{f * g\}(s) = F(s)G(s)$,

or, equivalently,

(9) $\mathscr{L}^{-1}\{F(s)G(s)\}(t) = (f * g)(t)$.

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PROBLEM 1

(a.) $\mathcal{L}\{te^{-t}-3\} = \mathcal{L}\{te^{-t}\} - \mathcal{L}\{3\}$

$$= \boxed{\frac{1}{(s+1)^2} - \frac{3}{s}}$$

(b.) $\mathcal{L}\{13e^{2t}\sin(t+\pi)\} = \mathcal{L}\{13e^{2t}(\sin t\cos\pi + \sin\pi\cos t)\}$

$$= \mathcal{L}\{-13e^{2t}\sin t\}$$

$$= \boxed{-13\left(\frac{1}{(s-2)^2+1}\right)}$$

(c.) $\mathcal{L}\{te^{-2t}\sin(3t)\} = -\frac{d}{ds}\left(\mathcal{L}\{e^{-2t}\sin 3t\}(s)\right)$ using Table 7.2

$$= -\frac{d}{ds}\left(\frac{3}{(s+2)^2+9}\right)$$

$$= +3\left[(s+2)^2+9\right]^{-2} 2(s+2)$$

$$= \boxed{\frac{6(s+2)}{\left[(s+2)^2+9\right]^2}}$$

(d.) $f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ t^2 & 1 \le t \le 2 \\ \sin t & 2 < t \end{cases} = [u(t)-u(t-1)] + [u(t-1)-u(t-2)]t^2 + u(t-2)\sin t$

$$f(t) = \underset{\text{I}}{u(t)} + \underset{\text{II}}{[t^2-1]u(t-1)} + \underset{\text{III}}{[\sin t - t^2]u(t-2)}$$

I $\mathcal{L}\{u(t)\} = \frac{1}{s}$

II Use $\mathcal{L}\{g(t)u(t-1)\}(s) = e^{-s}\mathcal{L}\{g(t+1)\}(s)$. Identify that $g(t) = t^2-1 \quad \therefore\ g(t+1) = (t+1)^2-1 = t^2+2t+1-1$

$$\mathcal{L}\{g(t+1)\}(s) = \mathcal{L}\{t^2+2t\}(s) = \frac{2}{s^3} + \frac{2}{s^2}$$

$$\therefore\ \mathcal{L}\{(t^2-1)u(t-1)\}(s) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2}\right)$$

PROBLEM 1

(d.) III Again use $\mathcal{L}\{g(t)\,u(t-2)\}(s) = e^{-2s}\,\mathcal{L}\{g(t+2)\}(s)$

identitify that $g(t) = \sin t - t^2$ thus

$$\begin{aligned} g(t+2) &= \sin(t+2) - (t+2)^2 \\ &= \sin t \cos 2 + \sin 2 \cos t - t^2 + 2t - 4 \end{aligned}$$

$$\mathcal{L}\{g(t+2)\}(s) = \frac{\cos(2)}{s^2+1} + \frac{s\sin(2)}{s^2+1} - \frac{2}{s^3} + \frac{2}{s^2} - \frac{4}{s}$$

$$\therefore\ \mathcal{L}\{(\sin t - t^2)u(t-2)\}(s) = e^{-2s}\left(\frac{\cos(2)}{s^2+1} + \sin(2)\frac{s}{s^2+1} - \frac{2}{s^3} + \frac{2}{s^2} - \frac{4}{s}\right)$$

Thus putting it together,

$$\boxed{F(s) = \frac{1}{s} + e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2}\right) + e^{-2s}\left(\frac{\cos(2)}{s^2+1} + \sin(2)\frac{s}{s^2+1} - \frac{2}{s^3} + \frac{2}{s^2} - \frac{4}{s}\right)}$$

(e.)
$$\begin{aligned} \sin(t)\cos^2(t) &= \frac{1}{2i}\left(e^{it}-e^{-it}\right)\left[\frac{1}{2}\left(e^{it}+e^{-it}\right)\right]^2 \\ &= \frac{1}{8i}\left(e^{it}-e^{-it}\right)\left(e^{2it}+2+e^{-2it}\right) \\ &= \frac{1}{8i}\left(e^{3it}+2e^{it}+e^{-it}-e^{it}-2e^{-it}-e^{-3it}\right) \\ &= \frac{1}{8i}\left(e^{3it}-e^{-3it}\right) + \frac{1}{8i}\left(e^{it}-e^{-it}\right) \\ &= \frac{1}{4}\frac{1}{2i}\left(e^{3it}-e^{-3it}\right) + \frac{1}{4}\frac{1}{2i}\left(e^{it}-e^{-it}\right) \\ &= \frac{1}{4}\sin(3t) + \frac{1}{4}\sin(t). \end{aligned}$$

$$\begin{aligned} \mathcal{L}\{\sin t\cos^2 t\} &= \mathcal{L}\left\{\frac{1}{4}\sin(3t) + \frac{1}{4}\sin t\right\} \\ &= \boxed{\frac{1}{4}\left(\frac{3}{s^2+9} + \frac{1}{s^2+1}\right)} \end{aligned}$$

PROBLEM 2

a.) $F(s) = \dfrac{s+1}{s^2-2s+5} = \dfrac{s+1}{(s-1)^2+4} = \dfrac{(s-1)+2}{(s-1)^2+4}$

$\therefore \quad \mathcal{L}^{-1}\{F\}(t) = \boxed{e^t\cos(2t) + e^t\sin(2t)}$

b.) $F(s) = \dfrac{7s^3-2s^2-3s+6}{s^3(s-2)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s^3} + \dfrac{D}{s-2}$

$7s^3-2s^2-3s+6 = As^2(s-2) + Bs(s-2) + C(s-2) + Ds^3$

$\underline{s=0}\quad 6 = -2C \quad \therefore\ C=-3$

$\underline{s=1}\quad 8 = -A - B + 3 + D \quad (*)$

$\underline{s=2}\quad 56-8-\cancel{6}+\cancel{6} = 8D \Rightarrow 48 = 8D \ \therefore\ D=6$

$\underline{s=-1}\quad -\cancel{7}-2+3+\cancel{6} = A(-3) - B(-3) - 3(-3) - \cancel{6}$

$0 = -3A + 3B + 3$

$\Rightarrow A = B+1$

$(*)$ becomes $\quad 8 = -B-1-B+3+6$

$0 = -2B \quad \therefore\ B=0 \quad \therefore\ A=1$

$\mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s} - \dfrac{3}{s^3} + \dfrac{6}{s-2}\right\}$ note $\mathcal{L}\left\{\dfrac{1}{s^{n+1}}\right\} = \dfrac{t^n}{n!}$

$= \boxed{1 - \frac{3}{2}t^2 + 6e^{2t}}$

PROBLEM 2

c.) unmodified.

$$F(s) = e^{-s}\left(\frac{4s+2}{s(s+1)}\right)$$

$$\frac{4s+2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$4s+2 = A(s+1) + Bs$$

$s|$ $4 = A + B$

Const.| $2 = A \Rightarrow B = 2$

$$G(s) = \frac{2}{s} + \frac{2}{s+1} \quad \therefore \quad g(t) = 2 + 2e^{-t}$$

$$\mathcal{L}^{-1}\{G(s)e^{-s}\} = g(t-1)\,u(t-1) \qquad \underline{\text{Th}^{m}\ (8)\ \text{eq}^{n}\ 6}$$

$$= \left(2 + 2e^{-(t-1)}\right)u(t-1)$$

$$= \boxed{2\left(1 + e^{-t+1}\right)u(t-1)}$$

PROBLEM 2

(c.) $\mathcal{L}^{-1}\{e^{-s}F(s)\}(t) = f(t-1)\,u(t-1)$

(modified) $F(s) = \dfrac{2s+2}{s(s+1)} = \dfrac{2\cancel{(s+1)}}{s\cancel{(s+1)}} = \dfrac{2}{s} \Rightarrow \underline{f(t) = 2}$

$\Rightarrow \underline{f(t-1) = 2}$

$\therefore \mathcal{L}^{-1}\left\{e^{-s}\left(\dfrac{4s+2}{s(s+1)}\right)\right\}(t) = \boxed{2u(t-1)}$

(d.) $F(s) = \dfrac{100s(s+2)}{(s^2+25)\left[(s+2)^2+16\right](s-1)}$

$= \left(\dfrac{s}{s^2+25}\right)\left(\dfrac{s+2}{(s+2)^2+16}\right)\left(\dfrac{100}{s-1}\right)$

$\quad\; G \qquad\qquad H \qquad\qquad K$

$\mathcal{L}^{-1}\{F\} = \mathcal{L}^{-1}\{GHK\}$

$= g * h * k$

$= \boxed{\sin(5t) * e^{-2t}\cos 4t * 100e^{t}}$

PROBLEM 3

(a.) $y'' + y = 2e^t \qquad y(0) = 1\ , \ y'(0) = 2$

$$s^2 Y - s - 2 + Y = \frac{2}{s-1}$$

$$(s^2+1)Y = s + 2 + \frac{2}{s-1}$$

$$Y = \frac{s+2}{s^2+1} + \underline{\frac{2}{(s^2+1)(s-1)}}$$

nontrivial, need to do partial fractions to break it up.

$$\frac{2}{(s^2+1)(s-1)} = \frac{As+B}{s^2+1} + \frac{C}{s-1}$$

$$2 = (As+B)(s-1) + C(s^2+1)$$

$$2 = s^2(A+C) + s(B-A) + C - B$$

$\underline{s^2}|\quad 0 = A + C \quad \rightarrow \quad \underline{A = -C}$

$\underline{s}|\quad 0 = B - A \quad \rightarrow \quad A = B = -C$

$\underline{\text{const}}|\quad 2 = C - B \quad \rightarrow \quad 2 = C-(-C) = 2C \ \therefore\ C = 1 \Rightarrow A = B = -1$

Therefore,

$$Y = \frac{\cancel{s}}{s^2+1} + \frac{2}{s^2+1} + \underbrace{\left(\frac{-1\cancel{s}-1}{s^2+1}\right) + \frac{1}{s-1}}_{\text{from the par. frac.}}$$

$$= \frac{1}{s^2+1} + \frac{1}{s-1}$$

Taking $\mathcal{L}^{-1}$ yields

$$\therefore \boxed{y = \sin t + e^t}$$

PROBLEM 3 Continued

(b.) $y'' - 2y' + y = u(t-1) \qquad y(0) = 0 \;, \; y'(0) = 1$

$$s^2 Y - 1 - 2sY + Y = e^{-s}/s$$

$$(s^2 - 2s + 1) Y = 1 + \frac{1}{s} e^{-s}$$

$$Y = \underbrace{\frac{1}{s^2 - 2s + 1}}_{\text{II}} + \boxed{\underbrace{\frac{1}{s(s^2 - 2s + 1)}}_{F(s)} e^{-s}} \leftarrow \text{think. Th}^{\underline{m}} (8)$$

I

I

$$\frac{1}{s\underbrace{(s^2-2s+1)}_{(s-1)^2}} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$1 = A(s-1)^2 + B(s-1)s + Cs$$

$\underline{s=1}|$ $1 = C$

$\underline{s=0}|$ $1 = A$

$\underline{s=2}|$ $1 = A + 2B + 2C = 1 + 2B + 2$

$2B = -2 \quad \therefore \quad B = -1$

$$F(s) = \frac{1}{s} - \frac{1}{s-1} + \frac{1}{(s-1)^2}$$

$$f(t) = 1 - e^t + te^t$$

$$\mathcal{L}^{-1}\{F(s)e^{-s}\} = f(t-1)\,u(t-1) = \left(1 - e^{t-1} + (t-1)e^{t-1}\right)u(t-1)$$

II $\quad \dfrac{1}{s^2 - 2s + 1} = \dfrac{1}{(s-1)^2} \longrightarrow \mathcal{L}^{-1}\left\{\dfrac{1}{s^2-2s+1}\right\} = te^t.$

$$\boxed{Y(t) = te^t + \left(1 - e^{t-1}(t-2)\right)u(t-1)}$$

PROBLEM 3

(c) $w'' + w = \delta(t-\pi) \qquad w(0) = 0, \quad w'(0) = 0$

$$s^2 W + W = e^{-\pi s}$$

$$W = \underbrace{\frac{1}{s^2+1}}_{F(s)} e^{-\pi s}$$

$\longrightarrow f(t) = \sin t$

$\longrightarrow f(t-\pi) = \sin(t-\pi)$

$= \sin t \cos(-\pi) + \sin(-\pi)\cos t)$

$= -\sin t.$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1} e^{-\pi s}\right\}(t) = f(t-\pi)\, u(t-\pi) \qquad \text{Th}^{m} (8)$$

$$= -\sin t \; u(t-\pi)$$

$$\therefore \; w = \mathcal{L}^{-1}\{W\} = \boxed{-\sin t \; u(t-\pi)}$$

PROBLEM 4

$$\mathcal{L}\{\delta(t-a)\}(s) = \int_0^\infty e^{-st}\, \delta(t-a)\, dt$$

$$= e^{-st}\Big|_{t=a}$$

$$= e^{-sa}.$$

by def^{n} of DIRAC DELTA FNCT. assuming $a \geq 0$ so that $t=a$ is in fact in the integration region.