YOUR NAME HERE:

MA 341-004, Introduction to Differential Equations

Instructor: James Cook
Test II: Laplace Transforms
Date: Friday, June 8, 2007

Directions: Show your work, if you doubt that you've shown enough detail then ask.

1. (15 pts) Calculate the Laplace transforms of the following functions using the table of basic Laplace transforms plus possibly the given Theorems. Derive any needed trigonometric identities via the imaginary exponential tricks.

   (a.) $f(t) = \sin(t)\cos(2t)$

   (b.) $f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ t^2 & 1 \le t \le 2 \\ 4\sin(t) & t > 2 \end{cases}$.

   (c.) $f(t) = te^{-2t} + t\sin(t)$

2. (10pts) Prove two of the following items. Your choice, except you may not choose (iii.) and (v.). All of the proofs involve the definition of the Laplace transform. Copying the statement from the formula sheet is not a valid solution. We assume that $a, b \in \mathbb{R}$ and $a \ge 0$.

   (i.)
   $$\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f(t)\}(s-a)$$

   (ii.)
   $$\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$$

   (iii.)
   $$\mathcal{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s}$$

   (iv.)
   $$\mathcal{L}\{y'\}(s) = \cancel{sY(s) - y(0)} = s\,\bar{Y}(s) - y(0)$$

   (v.)
   $$\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)$$

   (vi.)
   $$\mathcal{L}\{e^{at}\sin(bt)\}(s) = \frac{b}{(s-a)^2+b^2}$$

3. (25 pts) Compute the inverse Laplace transforms of,

(a.)

$$F(s) = \frac{3s+9}{s^2-8s+7}$$

(b.)

$$F(s) = e^{-3s}\frac{3s+9}{s^2-8s+7}$$

(c.)

$$F(s) = \frac{10s^2+10s+10}{(s^2+4s+5)s}$$

(d.)

$$F(s) = \frac{s(s+2)}{(s^2+25)[(s+2)^2+16](s-1)}$$

please state your answer for (d.) in terms of a convolution product, you should not attempt the integrals involved.

4. (30 pts) Solve the following differential equations with the given initial conditions by the method of Laplace transforms.

(a.) $y'' + y' - 2y = 0$ where $y(0) = 2$ and $y'(0) = 1$

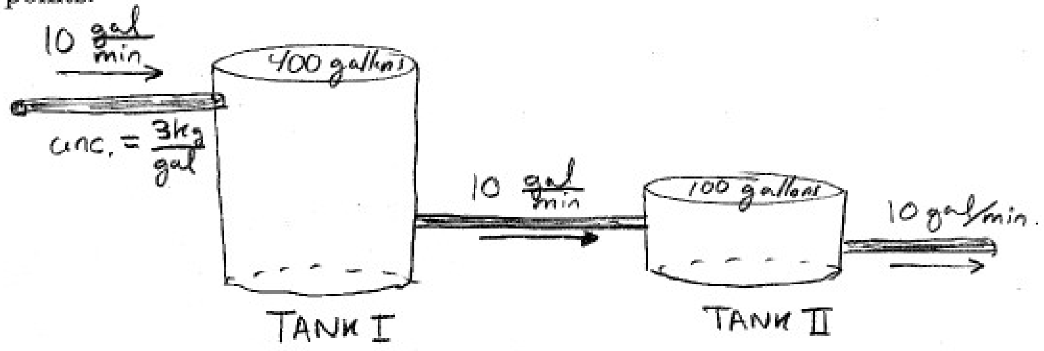
(b.) $y'' - 8y' + 7y = \delta(t-2)$ where $y(0) = 0$ and $y'(0) = 0$

(c.) $y'' - 8y' + 7y = u(t-2)$ where $y(0) = 0$ and $y'(0) = 0$

BONUS: (5 pts) We can see that the Laplace transform of e^{at} is the connected to the Laplace transforms of $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$ in a natural way. Let me guide you through this,
(i.) assume that $\mathcal{L}\{e^{zt}\}(s) = 1/(s-z)$ even in the case $z = \alpha + i\beta$
(ii.) rewrite $1/(s-(\alpha+i\beta))$ by muliplying by $\cancel{1 = [s+(\alpha+i\beta)]/[s+(\alpha+i\beta)]}$. This should make the imaginary unit disappear from the denominator and appear only in the numerator.
(iii.) recall that $e^{(\alpha+i\beta)t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$, now can you see both the Laplace transform of $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$ appear by equating the real and imaginary pieces.

$$1 = \frac{S-\alpha+i\beta}{S-\alpha+i\beta}$$

5. (10pts) TAKEHOME PROBLEM I: Suppose we have two tanks containing a mixture of salt and water. Tank I holds 400 gallons of water and at time zero has 4 kg of salt mixed uniformly in the water. Also, suppose that the second tank hold 100 gallons and has pure water at the beginning. Salty water flows into Tank I at a rate of 10 gallons per minute with a concentration of 3kg per gallon. Then a second pipe transfers water from Tank I to Tank II at a rate of 10 gallons per minute. Finally, a third pipe allows water to drain water from Tank II at a rate of 10 gallons per minute. Let x(t) and y(t) be the kgs of salt in Tank I and Tank II at time t. Find x(t) and y(t) via the method of Laplace. I have drawn a picture to illustrate the problem. You may puchase the differential equations that model this mixing tank problem for 2 points.



6. (10pts) TAKEHOME PROBLEM II, due Monday June 11. Solve the following differential equation by the method of Laplace transforms.

$$y'' + 5y' + 6y = e^t$$

you are given that y(3)=1 and y'(3)=0.

- You may place your sol^n in the envelope outside Ha 219 (DR. FULP's OFFICE) anytime before 4pm.

Test II sol^{n}, ma 341-004

(1a) $\sin(t)\cos(2t) = \frac{1}{2i}\left(e^{it} - e^{-it}\right)\frac{1}{2}\left(e^{2it} + e^{-2it}\right)$

$= \frac{1}{2}\frac{1}{2i}\left(e^{3it} + e^{-it} - e^{it} - e^{-3it}\right)$

$= \frac{1}{2}\frac{1}{2i}\left(e^{3it} - e^{-3it}\right) - \frac{1}{2}\frac{1}{2i}\left(e^{it} - e^{-it}\right)$

$= \frac{1}{2}\sin(3t) - \frac{1}{2}\sin(t)$

$\mathcal{L}\{\sin t\cos 2t\}(s) = \frac{1}{2}\mathcal{L}\{\sin 3t\}(s) - \frac{1}{2}\mathcal{L}\{\sin t\}(s)$

$= \boxed{\frac{1}{2}\frac{3}{s^2+9} - \frac{1}{2}\frac{1}{s^2+1}}$

(1b) $f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ t^2 & 1 \le t \le 2 \\ 4\sin t & t > 2 \end{cases}$

(*) ok, so techincally these are not quite equal at $t=1$ and $t=2$ but the Laplace transform is unchanged.

$f(t) \overset{(*)}{=} [u(t) - u(t-1)] + t^2[u(t-1) - u(t-2)] + 4\sin t\, u(t-2)$

$f(t) = u(t) + (t^2-1)u(t-1) + (4\sin t - t^2)u(t-2)$

$\mathcal{L}\{f\}(s) = \frac{1}{s} + e^{-s}\mathcal{L}\{(t+1)^2 - 1\}(s) + e^{-2s}\mathcal{L}\{4\sin(t+2) - (t+2)^2\}(s)$

$= \frac{1}{s} + e^{-s}\mathcal{L}\{t^2+2t\}(s) + e^{-2s}\mathcal{L}\{4\cos(2)\sin t + 4\sin(2)\cos t - t^2 - 4t - 4\}(s)$

$= \boxed{\frac{1}{s} + e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2}\right) + e^{-2s}\left(\frac{4\cos 2}{s^2+1} + \frac{(4\sin(2))\cdot s}{s^2+1} - \frac{2}{s^3} - \frac{4}{s^2} - \frac{4}{s}\right)}$

(1c) $f(t) = te^{-2t} + t\sin t$

$\mathcal{L}\{f\} = \mathcal{L}\{te^{-2t}\}(s) + \mathcal{L}\{t\sin t\}(s)$

$= \frac{1}{(s+2)^2} - \frac{d}{ds}\left[\frac{1}{s^2+1}\right]$

$= \boxed{\frac{1}{(s+2)^2} + \frac{2s}{(s^2+1)^2}}$

2i **Prove:** $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f(t)\}(s-a)$

$$\mathcal{L}\{e^{at}f(t)\}(s) = \int_0^\infty e^{-st}e^{at}f(t)\,dt$$
$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt$$
$$= \mathcal{L}\{f(t)\}(s-a). //$$

2ii **Prove:** $\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$

$$\mathcal{L}\{\delta(t-a)\}(s) = \int_0^\infty \delta(t-a)e^{-st}\,dt$$
$$= e^{-st}\Big|_{t=a}$$
$$= e^{-as}. //$$

2iii **Prove:** $\mathcal{L}\{u(t-a)\}(s) = e^{-as}/s$

$$\mathcal{L}\{u(t-a)\}(s) = \int_0^\infty e^{-st}u(t-a)\,dt$$
$$= \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st}\cdot 1\,dt$$
$$= \frac{-1}{s}(\underbrace{e^{-s\infty}}_{0} - e^{-as})$$
$$= e^{-as}/s. //$$

2iv **Prove:** $\mathcal{L}\{\frac{dy}{dt}\}(s) = s\,Y(s) - y(0)$

$$\mathcal{L}\{\tfrac{dy}{dt}\}(s) = \int_0^\infty \frac{dy}{dt} e^{-st}\,dt \quad , \text{ notice } \frac{d}{dt}(ye^{-st}) = \frac{dy}{dt}e^{-st} - sye^{-st}$$

$$= \int_0^\infty \left[\frac{d}{dt}(ye^{-st}) + sye^{-st}\right] dt$$

$$= ye^{-st}\Big|_0^\infty + s\int_0^\infty ye^{-st}\,dt$$

$$= \underbrace{y(\infty)e^{-s\infty}}_{0} - y(0)e^0 + s\,Y(s)$$

$$= s\,Y(s) - y(0). //$$

2v **Prove:** $\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s).$

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

$u = t-a$
$du = dt$
$t = u+a$
$t=a \Rightarrow u = a-a=0$

$$= \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= e^{-as}\int_0^\infty e^{-su} f(u)\,du$$

$$= e^{-as}\,\mathcal{L}\{f\}(s)$$

$$= e^{-as}\,F(s). //$$

2vi see $\boxed{E3}$ on pgs. 41 → 42 then apply the result proved in part i., $\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{s^2+b^2}\Big|_{s-a} = \frac{b}{(s-a)^2+b^2}$ //

③a $F(s) = \dfrac{3s+9}{s^2-8s+7} = \dfrac{A}{s-1} + \dfrac{B}{s-7}$

where we notice $s^2 - 8s + 7 = (s-1)(s-7)$.

$$3s+9 = A(s-7) + B(s-1)$$

$\underline{s=7}$ $30 = B \cdot 6 \quad \therefore \quad \underline{B=5}$

$\underline{s=1}$ $12 = A \cdot (-6) \quad \therefore \quad \underline{A=-2}$

$$\mathcal{L}^{-1}\{F\}(t) = \mathcal{L}^{-1}\left\{ \frac{-2}{s-1} + \frac{5}{s-7} \right\}(t)$$

$$= \boxed{-2e^{t} + 5e^{7t} = f(t)}$$

③b $F(s) = e^{-3s} \underbrace{\boxed{\dfrac{3s+9}{s^2-8s+7}}}$

call this $G(s)$ then $g(t) = -2e^{t} + 5e^{7t}$ from part a.

$$\mathcal{L}^{-1}\{F\}(t) = \mathcal{L}^{-1}\{e^{-3s}G(s)\}(t)$$

$$= g(t-3)\,u(t-3)$$

$$= \boxed{\left[-2e^{t-3} + 5e^{7(t-3)}\right]u(t-3)}$$

3c

$$F(s) = \frac{10s^2 + 10s + 10}{(s^2 + 4s + 5)s} = \underbrace{\frac{A(s+2) + B}{(s+2)^2 + 1}}_{} + \frac{C}{s}$$

my set-up here creates less work later, but

$$\frac{\tilde{A}s + \tilde{B}}{s^2 + 4s + 5}$$

will also work fine.

$$10s^2 + 10s + 10 = (A(s+2) + B)s + C(s^2 + 4s + 5)$$

$$10s^2 + 10s + 10 = s^2(A + C) + s(2A + B + 4C) + 5C$$

s^2 | $10 = A + C$

s | $10 = 2A + B + 4C$

s^0 | $10 = 5C \Rightarrow C = 2$

$\Rightarrow A = 10 - 2 = 8 = A$

$\Rightarrow B = 10 - 16 - 8 = -14 = B$

$$\mathcal{L}^{-1}\{F\}(t) = \mathcal{L}^{-1}\left\{ 8 \cdot \frac{s+2}{(s+2)^2 + 1} - 14 \cdot \frac{1}{(s+2)^2 + 1} + \frac{2}{s} \right\}(t)$$

$$= \boxed{8e^{-2t}\cos(t) - 14e^{-2t}\sin(t) + 2 = f(t)}$$

3d $\mathcal{L}^{-1}\{F\}(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\cdot\frac{s+2}{(s+2)^2+16}\cdot\frac{1}{s-1}\right\}(t)$

$$= \boxed{\cos(5t) * e^{-2t}\cos(4t) * e^{t} = f(t)}$$

(there are other possible answers)

4a $y'' + y' - 2y = 0 \qquad y(0) = 2,\ y'(0) = 1$

$$s^2 Y - 2s - 1 + sY - 2 - 2Y = 0$$

$$(s^2 + s - 2)Y = 3 + 2s$$

$$Y = \frac{3+2s}{s^2+s-2} = \frac{A}{s+2} + \frac{B}{s-1}$$

$$3 + 2s = A(s-1) + B(s+2)$$

$\underline{s=1}$ $\quad 5 = 3B \quad \therefore\ B = 5/3$

$\underline{s=-2}$ $\quad -1 = -3A \quad \therefore\ A = 1/3$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{1/3}{s+2} + \frac{5/3}{s-1}\right\}(t)$$

$$\boxed{y(t) = \frac{1}{3}e^{-2t} + \frac{5}{3}e^{t}}$$

(46) $y'' - 8y' + 7y = \delta(t-2)$ $\qquad y(0) = 0 = y'(0)$ (7)

$$s^2 Y - 8sY + 7Y = e^{-2s}$$

$$Y = e^{-2s}\left(\frac{1}{s^2-8s+7}\right)$$

$$= e^{-2s}\left[\frac{A}{s-1} + \frac{B}{s-7}\right]$$

$$= e^{-2s}\underbrace{\left[\frac{-1/6}{s-1} + \frac{1/6}{s-7}\right]}_{F(s)}$$

$$\frac{1}{s^2-8s+7} = \frac{A}{s-1} + \frac{B}{s-7}$$

$$1 = A(s-7) + B(s-1)$$

$s=7)$ $1 = 6B$ $\quad B = 1/6$

$s=1)$ $1 = -6A$ $\quad A = -1/6$

$$F(s) \Rightarrow f(t) = -\frac{1}{6}e^{t} + \frac{1}{6}e^{7t}$$

$$\downarrow$$

$$y = \mathcal{L}^{-1}\{e^{-2s}F(s)\}$$

$$= f(t-2)\,u(t-2)$$

$$= \boxed{\left[-\frac{1}{6}e^{t-2} + \frac{1}{6}e^{7(t-2)}\right]u(t-2) = y}$$

4c $\quad y'' - 8y' + 7y = u(t-2) \qquad y(0) = 0 = y'(0)$ 8

$$(s^2 - 8s + 7)\,Y = e^{-2s}/s$$

$$Y = e^{-2s}\Big[\underbrace{\frac{1}{s}\cdot\frac{1}{(s-7)(s-1)}}_{F(s)}\Big]$$

$$F(s) = \frac{A}{s} + \frac{B}{s-7} + \frac{C}{s-1} = \frac{1}{s}\,\frac{1}{(s-7)(s-1)}$$

$$A(s-7)(s-1) + Bs(s-1) + Cs(s-7) = 1$$

$\underline{s=1}$ $\quad C(-6) = 1 \quad \therefore \quad C = -1/6$

$\underline{s=7}$ $\quad 7B(6) = 1 \quad \therefore \quad B = 1/42$

$\underline{s=0}$ $\quad 7A = 1 \quad \therefore \quad A = 1/7$

$$F(s) = \Big[\frac{1}{7s} + \frac{1}{42}\frac{1}{s-7} - \frac{1}{6}\frac{1}{s-1}\Big]$$

$$f(t) = \frac{1}{7} + \frac{1}{42}e^{7t} - \frac{1}{6}e^{t}$$

$$y(t) = \mathcal{L}^{-1}\{e^{-2s}F(s)\}(t)$$

$$= f(t-2)\,u(t-2)$$

$$= \boxed{\frac{1}{7} + \frac{1}{42}e^{7(t-2)} - \frac{1}{6}e^{t-2} = y}$$

BONUS: Show that the Laplace transforms of $e^{\alpha t}$, $e^{\alpha t}\cos\beta t$, $e^{\alpha t}\sin\beta t$ all arise from $\mathcal{L}\{e^{\lambda t}\}(s) = \frac{1}{s-\lambda}$ for $\lambda \in \mathbb{C}$, $\lambda = \alpha + i\beta$.

$$\mathcal{L}\{e^{\lambda t}\}(s) = \int_0^\infty e^{-st} e^{\lambda t}\, dt = \int_0^\infty e^{-(s-\lambda)t}\, dt$$

$$= \frac{-1}{s-\lambda}\left[e^{-(s-\lambda)\infty} - e^0\right]$$

$$= \frac{1}{s-\lambda}$$

: this calculation still works when λ is complex. To be sure about this take ma 513. (or read Churchill)

When $\lambda = \alpha \in \mathbb{R}$ clearly we get $\mathcal{L}\{e^{\alpha t}\}(s) = \frac{1}{s-\alpha}$.
When $\lambda = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$ we get both $e^{\alpha t}\cos\beta t$ and $e^{\alpha t}\sin\beta t$, they appear as the Re & Im piece respective. Let me show this,

$$\mathcal{L}\{e^{\lambda t}\}(s) = \frac{1}{s-(\alpha+i\beta)}$$

$$= \frac{1}{(s-\alpha)-i\beta}\left[\frac{(s-\alpha)+i\beta}{(s-\alpha)+i\beta}\right]$$

$$= \frac{(s-\alpha)+i\beta}{(s-\alpha)^2+\beta^2}$$

$$= \frac{s-\alpha}{(s-\alpha)^2+\beta^2} + i\,\frac{\beta}{(s-\alpha)^2+\beta^2} \qquad \text{I}$$

On the other hand $e^{\lambda t} = e^{(\alpha+i\beta)t} = e^{\alpha t}\cos\beta t + i e^{\alpha t}\sin\beta t$

$$\mathcal{L}\{e^{\lambda t}\}(s) = \mathcal{L}\{e^{\alpha t}\cos\beta t\}(s) + i\,\mathcal{L}\{e^{\alpha t}\sin\beta t\}(s) \qquad \text{II}$$

Comparing the Re I & Re II and Im I & Im II we discover again the formulas for the Laplace transforms of $e^{\alpha t}\cos\beta t$ & $e^{\alpha t}\sin\beta t$. Again we see $\mathbb{C}$ unifies many things.