YOUR NAME HERE: ______

MA 341-004, Introduction to Differential Equations

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Test I: n-th order linear ODEs
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Directions: Show your work, if you doubt that you've shown enough detail then ask. If you need more space to work a problem just ask me for some paper.

1. (10 pts) Suppose a cup of coffee cools according to Newton's Law of Cooling. Further suppose that at $t = 0$ the coffee is at $T = 160$ then after one minute the coffee cools to $T = 140$. If the room the coffee is cooling in has an ambient temperature of 100 then find the temperature at time t. If my daughter Hannah likes to steal my coffee once it cools to 90 degrees do I need to worry about her sneaky plans to sip my coffee when I'm not looking ?

$\frac{dT}{dt} = k(T-100)$, by Newton's Law of Cooling.

$$T' - kT = -100k$$

$$\lambda - k = 0 \Rightarrow \lambda = k \Rightarrow T_h = e^{kt}$$

Guess $T_p = A$ then $T_p' = 0$ & $-kA = -100k$

$$\therefore \underline{A = 100}$$

Hence, $T(t) = 100 + c_1 e^{kt}$

Now apply the data $T(0) = 160$ & $T(1) = 140$

$$T(0) = 160 = 100 + c_1 \Rightarrow \underline{c_1 = 60}$$

$$T(1) = 140 = 100 + 60e^k \Rightarrow \frac{40}{60} = \frac{2}{3} = e^k$$

$$\Rightarrow k = \ln(2/3).$$

Therefore,

$$\boxed{T(t) = 100 + 60e^{\ln(2/3)t} = 100 + 60\left(\tfrac{2}{3}\right)^t}$$

Mathematically, Hannah is no problem $T(t) \not\to 90$ ever. However, in practica Hannah will spill coffee of any temperature, so its never safe if $h <$ 1meter.

2. (10pts) Given that $\mu = exp(\int P\,dx)$ show that

$$y = \frac{1}{\mu}\int \mu Q\,dx$$

is a solution to $dy/dx + Py = Q$. Assume that P, Q are continuous.

$$\frac{dy}{dx} + Py = Q \qquad \text{multiply by } \mu = e^{\int P dx}$$

$$e^{\int P dx}\frac{dy}{dx} + e^{\int P dx} P y = e^{\int P dx} Q \qquad (\mathrm{I.})$$

But notice $\frac{d}{dx}(\mu) = \frac{d}{dx}\left(e^{\int P dx}\right) = e^{\int P dx}\frac{d}{dx}\int P dx = \mu P.$

So we can rewrite (I.) above

$$\underbrace{\mu\frac{dy}{dx} + \frac{d\mu}{dx}y}_{\frac{d}{dx}(\mu y)} = \mu Q$$

$$\frac{d}{dx}(\mu y) = \mu Q$$

Integrate both sides w.r.t. x,

$$\mu y = \int \mu Q\,dx$$

$$\therefore \boxed{y = \frac{1}{\mu}\int \mu Q\,dx}$$

<u>Remark:</u> alternatively we can just differentiate the given alleged sol$^{\underline{n}}$ and see if it works.

3. (10 pts) Find the solution to

$$(2x + y\cos(xy))dx + (x\cos(xy) - 2y)dy = 0$$

that passes through the point $(\pi/2, 1)$. Notice this is an exact equation.

Since this is an exact eq^n we know $\exists\ F = F(x, y)$ such that

$$\text{I} \quad \frac{\partial F}{\partial x} = 2x + y\cos(xy)$$

$$\text{II} \quad \frac{\partial F}{\partial y} = x\cos(xy) - 2y$$

integrate I w.r.t. x, hold y fixed,

$$F = \int \frac{\partial F}{\partial x} dx = \int [2x + y\cos(xy)]\, dx$$

$$\therefore \quad \underline{F = x^2 + \frac{\cancel{y}\sin(xy)}{\cancel{y}} + C_1(y)} \quad \text{III}$$

Now substitute III into II

$$\frac{\partial F}{\partial y} = \cancel{x\cos(xy)} + \frac{\partial C_1}{\partial y} = \cancel{x\cos(xy)} - 2y$$

$$\therefore \int \frac{dC_1}{dy} dy = \int -2y\, dy$$

$$\Rightarrow C_1 = -y^2 + C_2$$

So we find $\underline{F(x,y) = x^2 - y^2 + \sin(xy)}$

Sol^n's have the form $x^2 - y^2 + \sin(xy) = k$ the question that remains is what is k? we know $(\pi/2, 1)$ is on the sol^n so

$$\frac{\pi^2}{4} - \cancel{1} + \cancel{\sin}\left(\frac{\pi}{2}\right) = k \quad \Rightarrow \underline{k = \pi^2/4}$$

$$\therefore \quad \boxed{x^2 - y^2 + \sin(xy) = \pi^2/4}$$

4. (20 pts) Find the general solutions to the following differential equations,

(a.) $y'' + y' - 2y = 0$

$$\lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1) = 0 \quad \therefore \lambda_1 = -2, \ \lambda_1 = 1$$

$$\boxed{y = c_1 e^{-2x} + c_2 e^{x}}$$

(b.) $y'' + 4y' + 8y = 0$

$$\lambda^2 + 4\lambda + 8 = 0 \longrightarrow \lambda = \frac{-4 \pm \sqrt{16 - 32}}{2} = -2 \pm 2i$$

$$\alpha = -2, \ \beta = 2 \quad \therefore \quad \boxed{y = c_1 e^{-2x} \cos(2x) + c_2 e^{-2x} \sin(2x)}$$

(c.) $y''' + y' = 0$

$$\lambda^3 + \lambda = \lambda(\lambda^2 + 1) = 0 \qquad \lambda = 0 \ \ \& \ \ \lambda = \pm i$$

$$\therefore \quad \boxed{y = c_1 + c_2 \cos(x) + c_3 \sin(x)}$$

(d.) $y''' - 3y'' - y' + 3y = 0$ it is useful to notice that the polynomial

$$\lambda^3 - 3\lambda^2 - \lambda + 3\cancel{\lambda} = 0$$

has $\lambda = 1$ as one of its solutions. $\Rightarrow$ the polynomial above factors,

$$(\lambda - 1)(\lambda^2 + B\lambda + C) = \lambda^3 - 3\lambda^2 - \lambda + 3$$

$$\lambda^3 + B\lambda^2 + C\lambda - \lambda^2 - B\lambda - C = \lambda^3 - 3\lambda^2 - \lambda + 3$$

$$\lambda^3 + \lambda^2[B - 1] + \lambda[C - B] - C = \lambda^3 - 3\lambda^2 - \lambda + 3$$

$$\therefore \boxed{C = -3}$$

$$C - B = -1 \Rightarrow B = C + 1 = -3 + 1 = \boxed{-2 = B}$$

$$B - 1 = -3 \Rightarrow -2 - 1 = -3 \quad \text{consistent} ☺$$

So our aux eq^{n} is $(\lambda - 1)(\lambda^2 - 2\lambda - 3) = 0$

$$(\lambda - 1)(\lambda - 3)(\lambda + 1) = 0$$

$$\therefore \quad \boxed{y = c_1 e^{x} + c_2 e^{3x} + c_3 e^{-x}}$$

5. (10 pts) Find the general solution of

$$y'' + y' - 2y = e^x$$

notice part (a.) of the previous problem is useful here.

We know $y_h = c_1 e^{-2x} + c_2 e^x$ from 4a.

Overlap of e^x with $e^x \Rightarrow y_p = Axe^x$

$$y_p' = A(e^x + xe^x) = Ae^x(1+x)$$

$$y_p'' = A[e^x(1+x) + e^x] = Ae^x(2+x)$$

$$y_p'' + y_p' - 2y_p = e^x$$

$$Ae^x(2+x) + Ae^x(1+x) - 2Axe^x = e^x$$

$$3Ae^x + \cancel{2Ae^x x} - \cancel{2Axe^x} = e^x$$

$$\Rightarrow 3A = 1 \quad \therefore \quad A = 1/3$$

Hence,

$$\boxed{y = c_1 e^{-2x} + c_2 e^x + \frac{1}{3}xe^x}$$

6. (15 pts) Find the general solution of

$$y'' + y = 3x^2 + x + 1$$

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

$$\Rightarrow y_h = c_1 \cos(x) + c_2 \sin(x)$$

no overlap.

Guess $y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$$y_p'' + y_p = 3x^2 + x + 1$$

$$2A + Ax^2 + Bx + C = 3x^2 + x + 1$$

x^2 | $A = 3$

x | $B = 1$

1 | $2A + C = 1 \Rightarrow 6 + C = 1 \therefore \underline{C = -5}$

$$\underline{y_p = 3x^2 + x - 5}$$

Thus

$$\boxed{y = c_1 \cos(x) + c_2 \sin(x) + 3x^2 + x - 5}$$

7. (10 pts) Find the general solution of

$$y'' + y = \sec(x)$$

via the method of variation of parameters. Recall,

$$v_1 = \int \frac{-gy_1}{y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx}}\, dx \qquad v_2 = \int \frac{gy_2}{y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx}}\, dx .$$

Note $\lambda^2 + 1 = 0 \therefore \lambda = \pm i \Rightarrow y_1 = \cos(x),\ y_2 = \sin(x)$

$y_1 y_2' - y_2 y_1' = \cos^2 x + \sin^2 x = 1$ thus

$$V_1 = \int -\sec(x)\cos(x)dx = \int -dx = \underline{-x = V_1}$$

$$V_2 = \int \sec(x)\sin(x)\,dx$$

$$= \int \frac{\sin(x)dx}{\cos(x)} \qquad \text{let } \begin{aligned} u &= \cos(x) \\ du &= -\sin(x)dx \end{aligned}$$

$$= \int -\frac{du}{u}$$

$$= -\ln|u|$$

$$= -\ln|\cos(x)|$$

$$\therefore \boxed{y = c_1\cos(x) + c_2\sin(x) - x\cos(x) - \ln|\cos(x)|\sin(x)}$$

Remark: this sol$^{\underline{n}}$ is incorrect. Why?

8. (6 pts) Find the general solution of

$$y'' + y = \sec(x) + 3x^2 + x + 1$$

please use the previous two problems for the sake of time.

$y'' + y = 3x^2 + x + 1$ has $y_{p_1} = 3x^2 + x - 5$

$y'' + y = \sec(x)$ has $y_{p_2} = -x\cos(x) - \ln|\cos(x)|\sin(x)$

By superposition,

$$\boxed{y = c_1\cos(x) + c_2\sin(x) + 3x^2 + x - 5 - x\cos(x) - \ln|\cos x|\sin x}$$

9. (9 pts) Use the method of anihilators to find the correct form for the particular solution (don't find $A, B, C, \ldots$ just set it up) of the following differential equation, $D = d/dx$ as usual,

$$(D+1)(D^2+1)(D-3)^2[y](x) = e^{3x}$$

e^{3x} is anihilated by $D - 3 = A$.

$(D-3)(D+1)(D^2+1)(D-3)^2[y](x) = (D-3)e^{3x} = 0$

Corresponding homog. eq$^{\underline{n}}$ is

$(D+1)(D^2+1)(D-3)^3[y](x) = 0$ with sol$^{\underline{n}}$ 2

$y_h = c_1e^{-x} + c_2\cos(x) + c_3\sin(x) + c_4e^{3x} + c_5xe^{3x} + \underline{c_6x^2e^{3x}}$

suggests $\boxed{y_p = Ax^2e^{3x}}$ works. (extra part)

BONUS: (5 pts) Let $f(x) = Re\{\exp(ix)\}$ and let $g(x) = Im\{\exp(ix)\}$ prove that f and g are linearly independent.

$\exp(ix) = \cos(x) + i\sin(x)$

$f(x) = Re\{\exp(ix)\} = \cos(x)$

$g(x) = Im\{\exp(ix)\} = \sin(x)$

$c_1\cos(x) + c_2\sin(x) = 0 \xrightarrow[x=0]{} c_1\cos(0) + 0 = 0 \therefore \underline{c_1 = 0}$

(this calculation proves LI) $\xrightarrow[x=\pi/2]{} c_1(0) + c_2 = 0 \therefore \underline{c_2 = 0}$

So $c_1 = c_2 = 0$.