SOLUTION TO MA 341 EXAM I

PROBLEM 1 I'll show my work,

$$2y'' - y' + 5ty = te^t\cos t - e^{2t}$$

$5t$ ← not constant coefficient $\Rightarrow$ can't use undet. coeff. thus $\boxed{\text{NO}}$

$$y'' + 3y' - 4y = \underbrace{t\csc t}_{} - t^2e^t$$

→ can't use undet. coeff. thus $\boxed{\text{NO}}$

$y'' + y' + 2y = 2te^t - t^2e^{-t}$, no problem $\boxed{\text{YES}}$

PROBLEM 2

$\cos(x)$ and $\sin(x)$ are linearly <u>independent</u> $\boxed{\text{FALSE}}$

[if they were linearly dependent then we would be able to write $\cos(x)$ as a <u>constant</u> multiple of $\sin x$ $\}$ $\cos(x) = c\sin(x) \Rightarrow c = \frac{\cos(x)}{\sin(x)}$ (not a constant.)]

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e^x and xe^x are linearly <u>independent</u> $\boxed{\text{TRUE}}$

[Assume otherwise then $e^x = cxe^x \Rightarrow c = \frac{1}{x} \neq$ constant.]

$x^2y'' + xy' + y = \sin^2(x)$ is a linear O.D.E $\boxed{\text{TRUE}}$

all appear linearly.

$A = D^2 + 4$ has $A[\cos(x)] = D^2\cos(x) + 4\cos(x)$ $\boxed{\text{FALSE}}$

$$= -D\sin(x) + 4\cos(x)$$
$$= -\cos(x) + 4\cos(x) = 3\cos(x) \neq 0.$$

Suppose $A[f] = 0$ then $LA[f] = L[0] = 0$. $\boxed{\text{TRUE}}$

[admittably I'd rather have written LA instead of AL in general $AL \neq LA$ although for all the examples we've examined this was the case.

$L = \exp(x)\frac{d}{dx}$ $\quad A = D^2 + 1$ then $A\sin(x) = 0$.

But $AL(\sin x) = A[\exp(x)\cos(x)]$
$= D[e^x\cos x - e^x\sin x] + e^x\cos x$
$= e^x\cos x - e^x\sin x - e^x\sin x + e^x\cos x + e^x\cos x \neq 0$]

PROBLEM 2 continued

I meant for you to think of const. coeff. linear diff. operators (not like $e^x \frac{d}{dx} = L$) as we just saw $AL \neq LA$ in general.

If this issue confused you on the test let me know.

$$\underbrace{[2x + y\cos(xy)]}_{M}dx + \underbrace{[x\cos(xy) - 2y]}_{N}dy = 0$$

$$\frac{\partial M}{\partial y} = \cos(xy) - xy\sin(xy)$$

$$\frac{\partial N}{\partial x} = \cos(xy) - xy\sin(xy) \quad \therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \underline{\text{exact}}.$$

$\boxed{\text{TRUE}}$

PROBLEM 3 $\quad y'' - 2y' + 5y = te^{2t}\sin t$

$$\lambda^2 - 2\lambda + 5 = 0 \rightarrow \lambda = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm i$$

$$y_h = c_1e^t\cos(t) + c_2e^t\sin t$$

$y_1 = e^t\cos t$

$y_2 = e^t\sin t$

• Now I forgot that the $g(t)$ had e^{2t} not e^t when I spoke to some of you.

No overlap then so the naive guess will do,

$$\boxed{Y_p = (At + B)e^{2t}\sin t + (Ct + D)e^{2t}\sin t}$$

You can see this from our general formula, or the anihilator method, or the naive/overlap line of reasoning.

PROBLEM 3

$$(D^2+1)(D+3)^2 y = x^2e^{-3x} + \cos(x)$$

Let's use the anihilator method. First notice the homogeneous sol^n has the form:

$$y_h = c_1 \cos x + c_2 \sin x + c_3 e^{-3x} + c_4 x e^{-3x}$$

[this follows quickly since we know that $P(D)\, e^{\lambda x} = P(\lambda) e^{\lambda x}$, the polynomial in D is the same as the aux. eq^n. and

$(\lambda^2+1)(\lambda+3)^2 = 0$ has sol^n's $\lambda = \pm i$ & $\lambda = -3$ (twice)]

Now the anihilator of $x^2e^{-3x} + \cos x$ can be found by thinking about what DEq^n's these functions arise as sol^n's to.

x^2e^{-3x} is sol^n to $(D+3)^3 Y = 0$

$\cos x$ is sol^n to $(D^2+1)Y = 0$

Thus $A = (D+3)^3(D^2+1)$ will work. Now multiply by A our DEq^n:

$$(D^2+1)^2(D+3)^5 y = A(x^2e^{-3x} + \cos(x)) = 0$$

This "corresponding homogeneous eq^n" has the sol^n

$$y = c_1 \cos(x) + c_2 \sin(x) + c_3 e^{-3x} + c_4 x e^{-3x}$$
$$+ \underbrace{c_5 x\cos x + c_6 x \sin x + c_7 x^2 e^{-3x} + c_8 x^3 e^{-3x} + c_9 x^4 e^{-3x}}_{}$$

this is the correct choice for y_p.

- The other two methods would also have given this same results. Most of you need to think more about this.

PROBLEM 4 Assume $x > 0$

$$\frac{dy}{dx} = \frac{y}{x} + 2x + 1 \longrightarrow \frac{dy}{dx} - \left(\frac{1}{x}\right)y = 2x+1$$

Identify that $P(x) = -1/x$ and $Q(x) = 2x+1$. Calculate then the integrating factor, we use $|x| = x$ as $x>0$,

$$\mu = \exp\left(\int -\frac{1}{x}dx\right) = \exp(-\ln|x|) = \exp\left(\ln\left(\frac{1}{x}\right)\right) = \boxed{\frac{1}{x} = \mu}$$

Now multiply by $\mu = 1/x$,

$$\underbrace{\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y} = \frac{1}{x}(2x+1) = 2 + \frac{1}{x}$$

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = 2 + \frac{1}{x}$$

Integrate both sides,

$$\frac{1}{x}y = 2x + \ln|x| + C \quad , \text{ note } |x| = x \text{ again,}$$

$$\boxed{y = 2x^2 + x\ln(x) + Cx}$$

PROBLEM 5 Let m, k be constants with $m > 0$ and we'll leave k arbitrary for fun. Find the eq$^{\underline{n}}$ of motion for a system with Newton's Law,

$$m\frac{dV}{dt} = kV^3 \quad \& \quad \underline{x(0) = 0, \; V(0) = 10}$$

Seperate variables

$$\frac{dV}{V^3} = \frac{k}{m}dt$$

Integrate,

$$\frac{-1}{2V^2} = \frac{kt}{m} + C_1$$

$\boxed{\text{PROBLEM 5}}$ Continuing,

$$\frac{-1}{2V^2} = \frac{kt}{m} + C_1 \quad \Rightarrow \quad V^2 = \frac{-1}{2}\left(\frac{1}{kt/m + C_1}\right)$$

Then taking the squareroot,

$$V = \pm\sqrt{\frac{-1}{2}\left(\frac{1}{kt/m + C_1}\right)}$$

Wether it's (+) or (-) depends on the intial condition.
Here $V(0) = 10$ so we must choose (+)

$$10 = +\sqrt{\frac{-1}{2}\frac{1}{C_1}} \rightarrow 100 = \frac{-1}{2C_1}$$

$$\rightarrow \underline{C_1 = \frac{-1}{200}}$$

Thus we find $\boxed{V(t) = \sqrt{\frac{-1}{2}\left(\frac{1}{kt/m - \frac{1}{200}}\right)}}$

it turns out that
from the math we could take $k < 0$ or $k > 0$

$$V(t) = \sqrt{\frac{1}{\frac{1}{100} - \frac{2kt}{m}}} = \left(\frac{1}{100} - \frac{2kt}{m}\right)^{-1/2}$$

If $k < 0$ then domain (V) included all $t \geq 0$
If $k > 0$ then domain (V) includes all $t \leq 0$
Physically since it's friction $\Rightarrow k < 0$ so then the velocity makes sense for positive time which is comforting. As you can see for $k < 0$

$$V(t) \longrightarrow 0 \quad \text{as} \quad t \rightarrow \infty$$

I wouldn't have expected all this w/o some extra question I didn't ask.

PROBLEM 5 We could note $V = \frac{dx}{dt}$ then integrate directly but the following is more interesting & easy.

$$m \frac{dV}{dt} = kV^3$$

sneaky.

$$m \frac{dx}{dt}\frac{dV}{dx} = mV\frac{dV}{dx} = kV^3$$

$$\int \frac{m}{k}\frac{dV}{V^2} = \int dx$$

$$\frac{-m}{kV} + C_2 = X$$

Now when $t = 0$ we have $\left.\begin{array}{l} X(t=0) = 0 \\ V(t=0) = 10 \end{array}\right\} \Rightarrow \underline{V(X=0) = 10}$

$$0 = \frac{-m}{10k} + C_2 \quad \therefore \quad C_2 = \frac{m}{10k}$$

Thus,

$$X = \frac{m}{k}\left(\frac{1}{10} - \frac{1}{V}\right)$$

$$= \boxed{\frac{m}{k}\left(\frac{1}{10} - \sqrt{2\left(\frac{1}{200} - \frac{kt}{m}\right)}\right) = X(t)}$$

You can easily verify that $X(0) = 0$ as it ought.

Finally the acceleration can be gleaned from $m\frac{dV}{dt} = kV^3$,

$$a = \frac{dV}{dt} = \frac{k}{m}V^3 = \boxed{\frac{k}{m}\left(\frac{1}{100} - \frac{2kt}{m}\right)^{-3/2} = a(t)}$$

$\boxed{\text{PROBLEM 6}}$ Given $y(0) = 3$ and $y'(0) = 0$, solve

$$y'' + 2y' - 8y = 0$$

$$\lambda^2 + 2\lambda - 8 = (\lambda + 4)(\lambda - 2) = 0 \quad \therefore \underline{\lambda_1 = -4, \ \lambda_2 = 2}$$

$$y(x) = c_1 e^{-4x} + c_2 e^{2x}$$ (could use t just as well here)

$$y'(x) = -4c_1 e^{-4x} + 2c_2 e^{2x}$$

$$y(0) = c_1 + c_2 = 3 \longrightarrow -2c_1 - \cancel{2c_2} = -6$$

$$y'(0) = -4c_1 + 2c_2 = 0 \qquad -4c_1 + \cancel{2c_2} = 0$$

$$-6c_1 = -6$$

$$\therefore c_1 = 1$$

$$c_2 = 3 - c_1 = 3 - 1 = 2 = c_2$$

$$\therefore \boxed{y = e^{-4x} + 2e^{2x}}$$

PROBLEM 7

$$y'' + 4y = \tan(2x)$$

$$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i \Rightarrow Y_1 = \cos 2x, \quad Y_2 = \sin 2x$$

Note, $W[Y_1, Y_2] = Y_1 Y_2' - Y_2 Y_1'$

$$= (\cos 2x)(2\cos 2x) - (\sin 2x)(-2\sin 2x)$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$$= 2$$

Use eq^{n} (10),

$$V_1 = \int \frac{-\tan(2x)\sin(2x)}{Y_1 Y_2' - Y_2 Y_1'}\,dx = \int \frac{-\sin^2 2x}{2\cos 2x}\,dx$$

$$= \frac{-1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\,dx$$

$$= \frac{-1}{2}\int \sec(2x) + \frac{1}{2}\int \cos(2x)\,dx$$

$$= \frac{-1}{4}\ln|\sec(2x)+\tan(2x)| + \frac{1}{4}\sin(2x).$$

$$V_2 = \int \frac{\tan(2x)\cos(2x)}{2}\,dx = \int \frac{\sin(2x)}{2}\,dx$$

$$= \frac{-1}{4}\cos(2x)$$

Recall that $y_p = y_1 V_1 + y_2 V_2$ so assemble the general sol^{n},

$$\boxed{y = c_1\cos(2x) + c_2\sin(2x) - \frac{1}{4}\cos(2x)\ln|\sec(2x)+\tan(2x)|}$$

$$+ \frac{1}{4}\cancel{\cos(2x)\sin(2x)} - \frac{1}{4}\cancel{\sin(2x)\cos(2x)}$$

PROBLEM 8 We know from #7 that

$y_1 = -\frac{1}{4}\cos(2x)\ln|\sec 2x + \tan 2x|$ is a particular sol^n

for $y'' + 4y = \tan(2x)$. Find y_{p_2} a particular sol^n for $y'' + 4y = x^3 + x^2 + 2$ then we'll be able to construct $y_p = y_{p_1} + y_{p_2}$ the particular sol^n for $y'' + 4y = x^3 + x^2 + 2 + \tan(2x)$. Let's find Y_{p_2} as we already have Y_{p_1}. Notice we can use undet. coeff and there is no over lap as $Y_1 = \cos 2x$ & $Y_2 = \sin 2x$.

$$Y_{p_2} = Ax^3 + Bx^2 + Cx + D$$

$$Y_{p_2}' = 3Ax^2 + 2Bx + C$$

$$Y_{p_2}'' = 6Ax + 2B$$

Now substitute into $y_{p_2}'' + 4y_{p_2} = x^3 + x^2 + 2$,

$$(6Ax + 2B) + 4Ax^3 + 4Bx^2 + 4Cx + 4D = x^3 + x^2 + 2$$

$$4A = 1 \quad \rightarrow \quad A = 1/4$$

$$4B = 1 \quad \rightarrow \quad B = 1/4$$

$$6A + 4C = 0 \quad \rightarrow \quad C = -\tfrac{1}{4}(6A) = -\tfrac{3}{2}\left(\tfrac{1}{4}\right) = -\tfrac{3}{8} = C$$

$$2B + 4D = 2 \quad \rightarrow \quad D = \tfrac{1}{4}(2 - 2B) = \tfrac{1}{4}\left(\tfrac{3}{2}\right) = \tfrac{3}{8} = D$$

Therefore, the general sol^n is (using superposition)

$$\boxed{y = c_1\cos 2x + c_2\sin 2x - \frac{1}{4}\cos 2x\ln|\sec 2x + \tan 2x| + \frac{1}{4}x^3 + \frac{1}{4}x^2 - \frac{3}{8}x + \frac{3}{8}}$$