MA 430, 12/12/06
FINAL EXAM

1. Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that $\det(A) = ad - bc$

$$Ae_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = ae_1 + ce_2$$
$$Ae_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = be_1 + de_2$$

$$Ae_1 \wedge Ae_2 = (ae_1 + ce_2) \wedge (be_1 + de_2)$$
$$= ab\underbrace{e_1 \wedge e_1}_{=0} + ade_1 \wedge e_2 + cbe_2 \wedge e_1 + cd\underbrace{e_2 \wedge e_2}_{=0}$$

$$= ad\, e_1 \wedge e_2 - bc\, e_1 \wedge e_2$$
$$= (ad - bc)\, e_1 \wedge e_2$$
$$= \det(A)\, e_1 \wedge e_2$$

$\therefore \det(A) = ad - bc$ ✓

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2. Show $*dz = dx \wedge dy$

✓ $$*dz = \frac{1}{p!}\frac{1}{(n-p)!}\alpha^i \epsilon_{ijk}\, dx^j \wedge dx^k$$
$$= \frac{1}{1!}\frac{1}{(3-1)!}\delta^{3i}\epsilon_{ijk}\, dx^j \wedge dx^k$$
$$= \frac{1}{2}\epsilon_{3jk}\, dx^j \wedge dx^k$$
$$= \frac{1}{2}\left(\epsilon_{312}\, dx \wedge dy + \epsilon_{321}\, dy \wedge dx\right)$$
$$= \frac{1}{2}\left(\epsilon_{312}\, dx \wedge dy + (-\epsilon_{312})(-dx \wedge dy)\right)$$
$$= \frac{1}{2}\left(2\epsilon_{312}\, dx \wedge dy\right)$$ (Note: $\epsilon_{312} = \epsilon_{123} = 1$)
$$= dx \wedge dy$$

3. Show $\omega_A \wedge \omega_B = \Phi_{A\times B}$

$$\omega_A = a\,dx + b\,dy + c\,dz \quad , \quad \omega_B = f\,dx + g\,dy + h\,dz$$

$$\begin{aligned}\omega_A \wedge \omega_B &= (a\,dx + b\,dy + c\,dz) \wedge (f\,dx + g\,dy + h\,dz) \\ &= ag\,dx\wedge dy + ah\,dx\wedge dz + bf\,dy\wedge dx + bh\,dy\wedge dz + cf\,dz\wedge dx + cg\,dz\wedge dy \\ &= (bh-cg)\,dy\wedge dz + (cf-ah)\,dz\wedge dx + (ag-bf)\,dx\wedge dy \\ &= \Phi_{A\times B}\end{aligned}$$

SINCE $A\times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ f & g & h \end{vmatrix} = (bh-cg)\hat{i} + (cf-ah)\hat{j} + (ag-bf)\hat{k}$

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4. (i) Show $df = \omega_{\nabla f}$

$$\begin{aligned} df &= \frac{\partial f}{\partial x_i}dx^i \\ &= (\nabla f)_i\,dx^i \\ &= \omega_{\nabla f}\end{aligned}$$

(ii) Show $d\Phi_G = (\nabla\cdot G)\,dx\wedge dy\wedge dz$

$$\Phi_G = G_1\,dy\wedge dz + G_2\,dz\wedge dx + G_3\,dx\wedge dy$$

$$\begin{aligned} d\Phi_G &= d(G_1\,dy\wedge dz + G_2\,dz\wedge dx + G_3\,dx\wedge dy) \\ &= \frac{\partial G_1}{\partial x}dx\wedge dy\wedge dz + \frac{\partial G_2}{\partial y}dy\wedge dz\wedge dx + \frac{\partial G_3}{\partial z}dz\wedge dx\wedge dy \\ &= \left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z}\right)dx\wedge dy\wedge dz \\ &= (\nabla\cdot G)\,dx\wedge dy\wedge dz\end{aligned}$$

nice

(my way was harder)

4. (cont'd.) (iii) Show $d\omega_F = \Phi_{\nabla\times F}$

$$\omega_F = F_1 dx + F_2 dy + F_3 dz$$

$$\begin{aligned}
d\omega_F &= d(F_1 dx + F_2 dy + F_3 dz) \\
&= \frac{\partial F_1}{\partial y} dy\wedge dx + \frac{\partial F_1}{\partial z} dz\wedge dx + \frac{\partial F_2}{\partial x} dx\wedge dy + \frac{\partial F_2}{\partial z} dz\wedge dy + \frac{\partial F_3}{\partial x} dx\wedge dz + \frac{\partial F_3}{\partial y} dy\wedge dz \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy\wedge dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz\wedge dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\wedge dy \\
&= \Phi_{\nabla\times F}
\end{aligned}$$

(iv) Show $d(d\alpha) = 0$ for any p-form α

$$\alpha = \frac{1}{p!}\alpha_{i_1\cdots i_p} dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

$$\begin{aligned}
d\alpha &= d\left(\frac{1}{p!}\alpha_{i_1\cdots i_p}\right) dx^{i_1}\wedge\cdots\wedge dx^{i_p} \\
&= \partial_m\left(\frac{1}{p!}\alpha_{i_1\cdots i_p}\right) dx^m\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}
\end{aligned}$$

$$\begin{aligned}
d(d\alpha) &= d\left(\partial_m\left(\frac{1}{p!}\alpha_{i_1\cdots i_p}\right)\right) dx^m\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p} \\
&= \underbrace{\partial_k (\partial_m}\left(\frac{1}{p!}\alpha_{i_1\cdots i_p}\right)) \underbrace{dx^k\wedge dx^m}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}
\end{aligned}$$

$\partial_k\partial_m$ is symmetric since $\partial_k\partial_m = \partial_m\partial_k$

However, $dx^k\wedge dx^m$ is antisymmetric since $dx^k\wedge dx^m = -dx^m\wedge dx^k$

Therefore, $d(d\alpha) = 0$.

5. Prove the following:

(i) $\int_S (\nabla \times F) \cdot dA = \int_{\partial S} F \cdot dl$

From 4(iii), we know that $d\omega_F = \Phi_{\nabla \times F}$

Thus $\int_S d\omega_F = \int_S \Phi_{\nabla \times F} = \int_S (\nabla \times F) \cdot dA$

Applying the Generalized Stoke's theorem, $\int_S d\omega_F = \int_{\partial S} \omega_F$

We were given that $\int_{\partial S} \omega_F = \int_{\partial S} F \cdot dl$

Therefore, $\int_S (\nabla \times F) \cdot dA = \int_S d\omega_F = \int_{\partial S} \omega_F = \int_{\partial S} F \cdot dl$.

So $\int_S (\nabla \times F) \cdot dA = \int_{\partial S} F \cdot dl$

5 (cont'd.) (ii) $\iiint_V (\nabla \cdot G)\,dx\,dy\,dz = \int_{\partial V} G \cdot dA$

From 4(ii), we know that $d\Phi_G = (\nabla \cdot G)\,dx \wedge dy \wedge dz$

Thus, $\int_V d\Phi_G = \int_V (\nabla \cdot G)\,dx \wedge dy \wedge dz$

We were given that $\int_V (\nabla \cdot G)\,dx \wedge dy \wedge dz = \iiint_V (\nabla \cdot G)\,dx\,dy\,dz$

Applying the Generalized Stoke's theorem, we have $\int_V d\Phi_G = \int_{\partial V} \Phi_G$

We know that $\int_{\partial V} \Phi_G = \int_{\partial V} G \cdot dA$

Therefore, $\iiint_V (\nabla \cdot G)\,dx\,dy\,dz = \int_V d\Phi_G = \int_{\partial V} \Phi_G = \int_{\partial V} G \cdot dA$.

So, $\iiint_V (\nabla \cdot G)\,dx\,dy\,dz = \int_{\partial V} G \cdot dA$

6. Show $*dt = -dx\wedge dy \wedge dz$

$$*dt = \frac{1}{p!}\frac{1}{(n-p)!}\gamma^{\mu}\epsilon_{\mu\nu\alpha\beta}\,dx^{\nu}\wedge dx^{\alpha}\wedge dx^{\beta}$$
$$= \frac{1}{1!}\frac{1}{(4-1)!}(-1)\delta^{0\mu}\epsilon_{\mu\nu\alpha\beta}\,dx^{\nu}\wedge dx^{\alpha}\wedge dx^{\beta}$$
$$= \frac{-1}{6}\epsilon_{0\nu\alpha\beta}\,dx^{\nu}\wedge dx^{\alpha}\wedge dx^{\beta}$$
$$= -\frac{1}{6}\big(\epsilon_{0123}dx\wedge dy\wedge dz + \epsilon_{0132}dx\wedge dz\wedge dy + \epsilon_{0213}dy\wedge dx\wedge dz$$
$$+ \epsilon_{0231}dy\wedge dz\wedge dx + \epsilon_{0312}dz\wedge dx\wedge dy + \epsilon_{0321}dz\wedge dy\wedge dx\big)$$
$$= -\frac{1}{6}\big(\epsilon_{0123}dx\wedge dy\wedge dz + (-\epsilon_{0123})(-dx\wedge dy\wedge dz) + (-\epsilon_{0123})(-dx\wedge dy\wedge dz)$$
$$+ \epsilon_{0123}dx\wedge dy\wedge dz + \epsilon_{0123}dx\wedge dy\wedge dz + (-\epsilon_{0123})(-dx\wedge dy\wedge dz)\big)$$
$$= -\frac{1}{6}\left(6\,\epsilon_{0123}dx\wedge dy\wedge dz\right)$$
$$= -dx\wedge dy\wedge dz \qquad \text{since } \epsilon_{0123} = 1.$$

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7. Given $J = -\rho dt + J_1 dx + J_2 dy + J_3 dz$,

(i) Show $*J = \rho\, dx\wedge dy\wedge dz - \Phi_J \wedge dt$

$$*J = -\rho *dt + J_1 *dx + J_2 *dy + J_3 *dz$$
$$= -\rho(-dx\wedge dy\wedge dz) + J_1(-dy\wedge dz\wedge dt) + J_2(-dz\wedge dx\wedge dt) + J_3(-dx\wedge dy\wedge dt)$$
$$= \rho\, dx\wedge dy\wedge dz - (J_1 dy\wedge dz + J_2 dz\wedge dx + J_3 dx\wedge dy)\wedge dt$$
$$= \rho\, dx\wedge dy\wedge dz - \Phi_{\vec{J}}\wedge dt$$

(ii) Show $d(*F) = *J \Rightarrow d(*J) = 0$

From problem 4(iv), I showed that $d(d\alpha) = 0$

In this case $*J = d(*F)$, so $d(*J) = d(d(*F))$

Since $d(d(*F)) = 0$, then $d(*J) = 0$ also.

6. Show $*dt = -dx\wedge dy\wedge dz$

$$*dt = \frac{1}{p!}\frac{1}{(n-p)!}\gamma^{\mu}\epsilon_{\mu\nu\alpha\beta}\,dx^{\nu}\wedge dx^{\alpha}\wedge dx^{\beta}$$
$$= \frac{1}{1!}\frac{1}{(4-1)!}(-1)\delta^{0\mu}\epsilon_{\mu\nu\alpha\beta}\,dx^{\nu}\wedge dx^{\alpha}\wedge dx^{\beta}$$
$$= \frac{-1}{6}\epsilon_{0\nu\alpha\beta}\,dx^{\nu}\wedge dx^{\alpha}\wedge dx^{\beta}$$
$$= -\frac{1}{6}\big(\epsilon_{0123}\,dx\wedge dy\wedge dz + \epsilon_{0132}\,dx\wedge dz\wedge dy + \epsilon_{0213}\,dy\wedge dx\wedge dz$$
$$+ \epsilon_{0231}\,dy\wedge dz\wedge dx + \epsilon_{0312}\,dz\wedge dx\wedge dy + \epsilon_{0321}\,dz\wedge dy\wedge dx\big)$$
$$= -\frac{1}{6}\big(\epsilon_{0123}\,dx\wedge dy\wedge dz + (-\epsilon_{0123})(-dx\wedge dy\wedge dz) + (-\epsilon_{0123})(-dx\wedge dy\wedge dz)$$
$$+ \epsilon_{0123}\,dx\wedge dy\wedge dz + \epsilon_{0123}\,dx\wedge dy\wedge dz + (-\epsilon_{0123})(-dx\wedge dy\wedge dz)\big)$$
$$= -\frac{1}{6}\left(6\,\epsilon_{0123}\,dx\wedge dy\wedge dz\right)$$
$$= -dx\wedge dy\wedge dz \qquad \text{since } \epsilon_{0123} = 1.$$

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7. Given $J = -\rho dt + J_1 dx + J_2 dy + J_3 dz$,

(i) Show $*J = \rho\, dx\wedge dy\wedge dz - \Phi_J \wedge dt$

$$*J = -\rho *dt + J_1 *dx + J_2 *dy + J_3 *dz$$
$$= -\rho(-dx\wedge dy\wedge dz) + J_1(-dy\wedge dz\wedge dt) + J_2(-dz\wedge dx\wedge dt) + J_3(-dx\wedge dy\wedge dt)$$
$$= \rho\, dx\wedge dy\wedge dz - (J_1\, dy\wedge dz + J_2\, dz\wedge dx + J_3\, dx\wedge dy)\wedge dt$$
$$= \rho\, dx\wedge dy\wedge dz - \Phi_J\wedge dt$$

(ii) Show $d(*F) = *J \Rightarrow d(*J) = 0$

From problem 4(iv), I showed that $d(d\alpha) = 0$

In this case $*J = d(*F)$, so $d(*J) = d(d(*F))$

Since $d(d(*F)) = 0$, then $d(*J) = 0$ also.

8. Is it true or false that...

$\cancel{\times}$ (i) $d(*\alpha) = *(d\alpha)$ $\rightarrow$ ~~True~~ false

$\alpha = x$

$*\alpha = x\,dx\wedge dy\wedge dz$

Let $\alpha = dx$, then $*dx = dy\wedge dz$ (in Euclidean)

and $d(dy\wedge dz) = d(dy)\wedge d(dz) = 0$

since $d(d\alpha) = 0$ from 4(iv).

$d\alpha = dx$

$\rightarrow *d\alpha = dy\wedge dz$

Now, start with $d(dx)$. Well, $d(dx) = 0$ also. BUT

So $d(*dx) = *(d(dx)) = 0$.

$d(*\alpha) = 0$.

$\checkmark$ (ii) $*(\alpha\wedge\beta) = *\alpha\wedge *\beta$ $\rightarrow$ False

Let $\alpha = dx$ & $\beta = dy$ (in Euclidean)

Then $*(dx\wedge dy) = dz$

and $*dx\wedge *dy = (dy\wedge dz)\wedge(dz\wedge dx) = 0$ due to $dz\wedge dz = 0$

$\times$ (iii) $\alpha\wedge\beta = -\beta\wedge\alpha$ $\rightarrow$ ~~True~~ false

let $\alpha = dx$ & $\beta = dy$

$dx\wedge dy = -dy\wedge dx$ (in Euclidean)

$dx\wedge dy = -dz\wedge dt = -dy\wedge dx$ (in Minkowski)

Counter Example

$(dx\wedge dy)\wedge(dt\wedge dz) = (dt\wedge dz)\wedge(dx\wedge dy)$

generally

$$\alpha_p\wedge\beta_q = (-1)^{pq}\beta_q\wedge\alpha_p$$

9. Given that $F = \omega_E \wedge dt + \Phi_B$, show that $*F = -\omega_B \wedge dt + \Phi_E$

$$F = \omega_E \wedge dt + \Phi_B$$
$$= E_x dx \wedge dt + E_y dy \wedge dt + E_z dz \wedge dt + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$

$$*F = E_x *(dx \wedge dt) + E_y *(dy \wedge dt) + E_z *(dz \wedge dt) + B_x *(dy \wedge dz) + B_y *(dz \wedge dx) + B_z *(dx \wedge dy)$$
$$= E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy + B_x(-dx \wedge dt) + B_y(-dy \wedge dt) + B_z(-dz \wedge dt)$$
$$= \Phi_E - (B_x dx + B_y dy + B_z dz) \wedge dt$$
$$= \Phi_E - \omega_B \wedge dt$$

OR

$$*F = -\omega_B \wedge dt + \Phi_E$$

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10. $F = \omega_E \wedge dt + \Phi_B$
$$= E_x dx \wedge dt + E_y dy \wedge dt + E_z dz \wedge dt + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$

$$dF = \frac{\partial E_x}{\partial y} dy \wedge dx \wedge dt + \frac{\partial E_x}{\partial z} dz \wedge dx \wedge dt + \frac{\partial E_y}{\partial x} dx \wedge dy \wedge dt + \frac{\partial E_y}{\partial z} dz \wedge dy \wedge dt$$
$$+ \frac{\partial E_z}{\partial x} dx \wedge dz \wedge dt + \frac{\partial E_z}{\partial y} dy \wedge dz \wedge dt + \frac{\partial B_x}{\partial x} dx \wedge dy \wedge dz + \frac{\partial B_x}{\partial t} dt \wedge dy \wedge dz$$
$$+ \frac{\partial B_y}{\partial y} dy \wedge dz \wedge dx + \frac{\partial B_y}{\partial t} dt \wedge dz \wedge dx + \frac{\partial B_z}{\partial z} dz \wedge dx \wedge dy + \frac{\partial B_z}{\partial t} dt \wedge dx \wedge dy$$
$$= \left[\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) dy \wedge dz + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) dz \wedge dx + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) dx \wedge dy\right] \wedge dt$$
$$+ \left(\frac{\partial B_x}{\partial t} dy \wedge dz + \frac{\partial B_y}{\partial t} dz \wedge dx + \frac{\partial B_z}{\partial t} dx \wedge dy\right) \wedge dt + \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) dx \wedge dy \wedge dz$$
$$= \Phi_{\nabla \times E} \wedge dt + \Phi_{\frac{\partial B}{\partial t}} \wedge dt + (\nabla \cdot B) dx \wedge dy \wedge dz$$
$$= \Phi_{\nabla \times E + \frac{\partial B}{\partial t}} \wedge dt + (\nabla \cdot B) dx \wedge dy \wedge dz$$

SINCE $dF = 0 \Rightarrow \nabla \times E + \frac{\partial B}{\partial t} = 0$ OR $\nabla \times E = -\frac{\partial B}{\partial t}$

$$\Rightarrow \nabla \cdot B = 0$$

10.(cont'd.) $*F = -\omega_B \wedge dt + \Phi_E$

$$= -B_x dx\wedge dt - B_y dy\wedge dt - B_z dz\wedge dt + E_x dy\wedge dz + E_y dz\wedge dx + E_z dx\wedge dy$$

$$\begin{aligned} d(*F) = &-\frac{\partial B_x}{\partial y} dy\wedge dx\wedge dt - \frac{\partial B_x}{\partial z} dz\wedge dx\wedge dt - \frac{\partial B_y}{\partial x} dx\wedge dy\wedge dt - \frac{\partial B_y}{\partial z} dz\wedge dy\wedge dt \\ &- \frac{\partial B_z}{\partial x} dx\wedge dz\wedge dt - \frac{\partial B_z}{\partial y} dy\wedge dz\wedge dt + \frac{\partial E_x}{\partial x} dx\wedge dy\wedge dz + \frac{\partial E_x}{\partial t} dt\wedge dy\wedge dz \\ &+ \frac{\partial E_y}{\partial y} dy\wedge dz\wedge dx + \frac{\partial E_y}{\partial t} dt\wedge dz + \frac{\partial E_z}{\partial z} dz\wedge dx\wedge dy + \frac{\partial E_z}{\partial t} dt\wedge dx\wedge dy \end{aligned}$$

$$\begin{aligned} = &-\left[\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) dy\wedge dz + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) dz\wedge dx + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) dx\wedge dy\right] \wedge dt \\ &+ \left(\frac{\partial E_x}{\partial t} dy\wedge dz + \frac{\partial E_y}{\partial t} dz\wedge dx + \frac{\partial E_z}{\partial t} dx\wedge dy\right) \wedge dt + \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) dx\wedge dy\wedge dz \end{aligned}$$

$$= -\Phi_{(\nabla\times B)} \wedge dt + \Phi_{\frac{\partial E}{\partial t}} \wedge dt + (\nabla\cdot E) dx\wedge dy\wedge dz$$

$$= -\Phi_{(\nabla\times B) - \frac{\partial E}{\partial t}} \wedge dt + (\nabla\cdot E) dx\wedge dy\wedge dz$$

Since $d(*F) = \mu_0 * J$, WHERE $\mu_0 * J = \underbrace{\mu_0 \rho}_{\rho/\epsilon_0} dx\wedge dy\wedge dz - \Phi_{\mu_0 J} \wedge dt$

then $\nabla\cdot E = \rho/\epsilon_0$

& $\nabla\times B - \frac{\partial E}{\partial t} = \mu_0 J \Rightarrow \nabla\times B = \mu_0 J + \frac{\partial E}{\partial t}$

$\left[\mu_0\epsilon_0 = \mu_0\left(\frac{1}{\mu_0}\right) = 1\right]$

OR

$$\nabla\times B = \mu_0 J + \mu_0\epsilon_0 \frac{\partial E}{\partial t}$$