Ma 430 Homework Sol'ns

PROBLEM 12

a.) We wish to prove $O(3) = \{A \in \mathbb{R}^{3\times 3} \mid A^TA = I\}$ is a group.
Note $I^TI = I$ $\therefore$ $I \in O(3)$. Let $A \in O(3)$ then we know $A^TA = I$ $\therefore$ $A^{-1} = A^T$ moreover

$$(A^{-1})^TA^{-1} = (A^T)^TA^{-1} = AA^{-1} = I \quad \therefore\ A^{-1} \in O(3).$$

Next suppose $A, B \in O(3)$ we know that $A^TA = I$ & $B^TB = I$;

$$\begin{aligned}(AB)^TAB &= B^TA^TAB \\ &= B^TIB \\ &= B^TB \\ &= I \quad \therefore\ (AB) \in O(3).\end{aligned}$$

Thus we have shown $O(3)$ has an identity element I, is closed under inverses and matrix multiplication. Finally as we know matrix multiplication is associative this proves $O(3)$ is a group.

(b.) We wish to prove $SO(3) = \{A \in \mathbb{R}^{3\times 3} \mid A^TA = I,\ \det A = 1\}$ is a group. Notice that

i.) $\det(I) = 1$ & $I^TI = I$. $\therefore$ $I \in SO(3)$

ii.) $A \in SO(3) \Rightarrow A^{-1} = A^T$ & $(A^{-1})^TA^{-1} = I$ (as before)
and $\det(A^{-1}) = \det(A^T) = \det A = 1$.
$\therefore$ $A^{-1} \in SO(3)$.

iii.) $A, B \in SO(3) \Rightarrow (AB)^TAB = I$ as in part. (a.)
$\det(AB) = \det A \det B = (1)(1) = 1$
$\therefore$ $(AB) \in SO(3)$.

Hence $SO(3)$ is a group.

PROBLEM 12

(C.) Let $\mathcal{J} = -I \in \mathbb{R}^{3\times3}$ and define the coset of $SO(3)$

$$SO(3)\mathcal{J} \equiv \{ B \in \mathbb{R}^{3\times3} \mid \exists A \in SO(3) \text{ s.t. } B = A\mathcal{J} \}$$

We show $SO(3) \cap SO(3)\mathcal{J} = \phi$. Let $B \in SO(3) \cap SO(3)\mathcal{J}$ then $B \in SO(3)$ & $B \in SO(3)\mathcal{J}$ $\therefore \exists A \in SO(3)$ s.t. $B = A\mathcal{J}$.

notice $\det(B) = 1$ & $\det(B) = \det(A\mathcal{J}) = \det(A)\det(\mathcal{J}) = -1$.
therefore there is no such $B \Rightarrow SO(3) \cap SO(3)\mathcal{J} = \phi$.

Now we show $O(3) = SO(3) \cup SO(3)\mathcal{J}$. To begin we show $O(3) \subset SO(3) \cup SO(3)\mathcal{J}$. Let $A \in O(3)$ then,

$$A^TA = I \quad \therefore \det(A^TA) = \det(A^T)\det(A) = \det(A)^2 = 1$$
$$\therefore \underline{\det(A) = \pm 1}.$$

If $\det(A) = 1 \Rightarrow A \in SO(3) \Rightarrow A \in SO(3) \cup SO(3)\mathcal{J}$.
If $\det(A) = -1$ then $\det(-A) = \det(-I)\det(A) = (-1)(-1) = 1$ thus $-A \in SO(3)$ and $A = (-A)(-I) \in SO(3)\mathcal{J}$. Thus $A \in SO(3) \cup SO(3)\mathcal{J}$. Hence $O(3) \subset SO(3) \cup SO(3)\mathcal{J}$.

Next suppose $A \in SO(3) \cup SO(3)\mathcal{J}$ then either $A \in SO(3)$ in which case $A^TA = I \therefore A \in O(3)$ or $A \in SO(3)\mathcal{J}$. (we know its not in both since $SO(3) \cap SO(3)\mathcal{J} = \phi$.)
As we were saying $A \in SO(3)\mathcal{J} \therefore \exists B \in SO(3)$ s.t. $A = B\mathcal{J}$ consider then that

$$\begin{aligned} A^TA &= (B\mathcal{J})^T(B\mathcal{J}) \\ &= \mathcal{J}^T B^T B \mathcal{J} \\ &= \mathcal{J}^T I \mathcal{J} \\ &= (-I)(I)(-I) \\ &= I \quad \therefore A \in O(3). \Rightarrow SO(3) \cup SO(3)\mathcal{J} \subset O(3). \end{aligned}$$

We conclude that $\boxed{O(3) = SO(3) \cup SO(3)\mathcal{J}}$

PROBLEM 13 Show $\mathcal{R} = \{\varphi : \mathbb{R}^3 \to \mathbb{R}^3 \mid \varphi(x) = Ax + b, A \in SO(3), b \in \mathbb{R}^3\}$ the set of rigid motions forms a group. Notice that $Id(x) = Ix + 0$ and $I \in SO(3), 0 \in \mathbb{R}^3 \therefore Id \in \mathcal{R}$, moreover this is the inverse for function composition since $Id \circ f = f \circ Id \quad \forall f : \mathbb{R}^3 \to \mathbb{R}^3$. Next let $\varphi_1, \varphi_2 \in \mathcal{R}$ so that $\exists A_1, A_2 \in SO(3)$ & $b_1, b_2 \in \mathbb{R}^3$ s.t.

$$\varphi_1(x) = A_1 x + b_1 \quad \& \quad \varphi_2(x) = A_2 x + b_2$$

$\forall x \in \mathbb{R}^3$. We show now that $\varphi_1 \circ \varphi_2 \in \mathcal{R}$,

$$\begin{aligned}(\varphi_1 \circ \varphi_2)(x) &= \varphi_1(\varphi_2(x)) \\ &= \varphi_1(A_2 x + b_2) \\ &= A_1(A_2 x + b_2) + b_1 \\ &= \underbrace{A_1 A_2}_{\in SO(3)} x + \underbrace{A_1 b_2 + b_1}_{\in \mathbb{R}^3} \Rightarrow \varphi_1 \circ \varphi_2 \in \mathcal{R}.\end{aligned}$$

Suppose $\varphi \in \mathcal{R}$, $\varphi(x) = Ax + b$ for $A \in SO(3)$ & $b \in \mathbb{R}^3$. We claim $\varphi^{-1}(x) = A^{-1}x - A^{-1}b$,

$$\begin{aligned}\varphi(\varphi^{-1}(x)) &= \varphi(A^{-1}x - A^{-1}b) \\ &= AA^{-1}x - AA^{-1}b + b \\ &= x\end{aligned}$$

Likewise $\varphi^{-1}(\varphi(x)) = x \quad \therefore \varphi^{-1}$ is the inverse of φ. moreover $A^{-1} \in SO(3)$ thus $\varphi^{-1} \in \mathcal{R}$. As we know fnct. composition is associative it follows $\mathcal{R}$ is a group.

PROBLEM 14 Throughout this problem $\det(A) = 1$.

$A \in O(3) \iff A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

$\Leftarrow$) Suppose $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $\det(A) = a^2+b^2 = 1$.

$$A^T A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix} = I \quad \therefore A \in O(3).$$

$\Rightarrow$) Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(3)$ and $\det(A) = ad - bc = 1$.

Notice that $A^T A = I \iff A^T = A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ thus

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \therefore \quad \begin{matrix} a = d & c = -b \\ b = -c & d = a \end{matrix}$$

Thus we find $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Next show $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $a^2+b^2 = 1 \iff A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

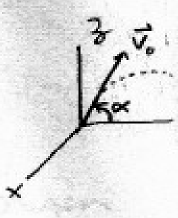
$\Leftarrow$) Let $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Choose $a = \cos\theta$ and $b = \sin\theta$

then $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $a^2+b^2 = \cos^2\theta + \sin^2\theta = 1$.

$\Rightarrow$) Given $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $a^2+b^2 = 1$ we may select θ such that $a = \cos\theta$ & $b = \sin\theta$ since $a^2+b^2 = 1$.

PROBLEM 15 Everyone did this nearly correct, might be a good test question...

III.1 Let the surface of the earth be represented by xy plane and assume that the acceleration due to gravity is $-g\hat{k}$. If a projectile of mass m is launched at a speed v_o from the origin at an angle α with horizontal, where $\vec{V}_o$ lies in yz plane.



Let γ denote trajectory of particle, with x, y, z coordinates. I leave off coordinate dependence because we are working in just one coord. set.

(a) $$\frac{d^2\gamma}{dt} \equiv \frac{dV}{dt} = \left(\frac{dV_x}{dt}, \frac{dV_y}{dt}, \frac{dV_z}{dt}\right) = (0, 0, -g)$$

$$\int_{V_o}^{V(t)} dV = V(t) - \vec{V}_o = \int_0^t (0,0,-g)\,d\tilde{t} = (0,0,-gt) \implies V(t) = (0,0,-gt) + \vec{V}_o$$

As $\vec{V}_o$ lies in yz plane, $\vec{V}_o = V_o(0, \cos\alpha, \sin\alpha)$ $\therefore$ $V(t) = (0, V_o\cos\alpha, V_o\sin\alpha - gt)$

$$\frac{d\gamma}{dt} \equiv V(t) \implies \int_{\gamma_o}^{\gamma(t)} d\gamma = \gamma(t) - \gamma_o = \int_0^t (0, V_o\cos\alpha, V_o\sin\alpha - g\tilde{t})\,d\tilde{t}, \quad \text{note } \gamma_o = \vec{0}$$

$$\therefore \boxed{\gamma(t) = \left(0, V_o t\cos\alpha, V_o t\sin\alpha - \tfrac{1}{2}gt^2\right)} \qquad \gamma(t) = (x, y, z).$$

(b) Max height occurs where $\frac{d}{dt}\left(V_o t\sin\alpha - \frac{1}{2}gt^2\right) = V_o\sin\alpha - gt = 0 \implies \boxed{t_{1/2} = \frac{V_o\sin\alpha}{g}}$

moreover we know this critical pt. to be a maximum as,

$$\frac{d^2}{dt^2}\left(V_o t\sin\alpha - \tfrac{1}{2}gt^2\right) = -g < 0 \quad \therefore \text{ this is a maxima for } z.$$

(c) The total time in flight is found by realizing that the flight ceases when the projectile hits the ground, that is $z(t) = z_o = 0$.

$$z(t) - z_o = 0 = V_o t\sin\alpha - \tfrac{1}{2}gt^2 \implies t_{total} = \frac{2V_o\sin\alpha}{g} \quad \left(\begin{array}{c} t=0, \text{ no motion} \\ \text{not sol}^{\text{n}} \text{ of interest} \end{array}\right)$$

The <u>RANGE</u> is THE DISTANCE TRAVERSED ALONG y in this case,

$$y(t_{total}) - y_o = y(t_{total}) = V_o(t_{total})\cos\alpha = V_o\left(\frac{2V_o\sin\alpha}{g}\right)\cos\alpha$$

$$\boxed{\text{Range} = \frac{2V_o^2\sin\alpha\cos\alpha}{g}} \quad \text{OR} \quad \boxed{\text{Range} = \frac{V_o^2\sin 2\alpha}{g}} \quad \text{by trig identity.}$$

$$\left(2\sin\alpha\cos\alpha = \frac{2}{2i}\left(e^{i\alpha} - e^{-i\alpha}\right)\frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right) = \frac{2}{2}\left(\frac{1}{2i}\left[e^{2i\alpha} - e^{-2i\alpha}\right]\right) = \sin 2\alpha.\right)$$

III.1d

(d) Show that the *maximum* Range(R) occurs at $\alpha = 45°$.

$$R = \frac{V_0^2 \sin 2\alpha}{g}$$

actually the range is the distance from the origin to the point of impact and so $R = -\frac{V_0^2 \sin(2\alpha)}{g}$ if $\sin(2\alpha) < 0$

$$\frac{dR}{d\alpha} = \frac{V_0^2}{g}\left(2\cos(2\alpha)\right) = 0 \Rightarrow \cos(2\alpha) = 0.$$

This probably accounts for the odd answer below as you are getting a minimum mainly because it is negative! its absolute value still maximizes the distance

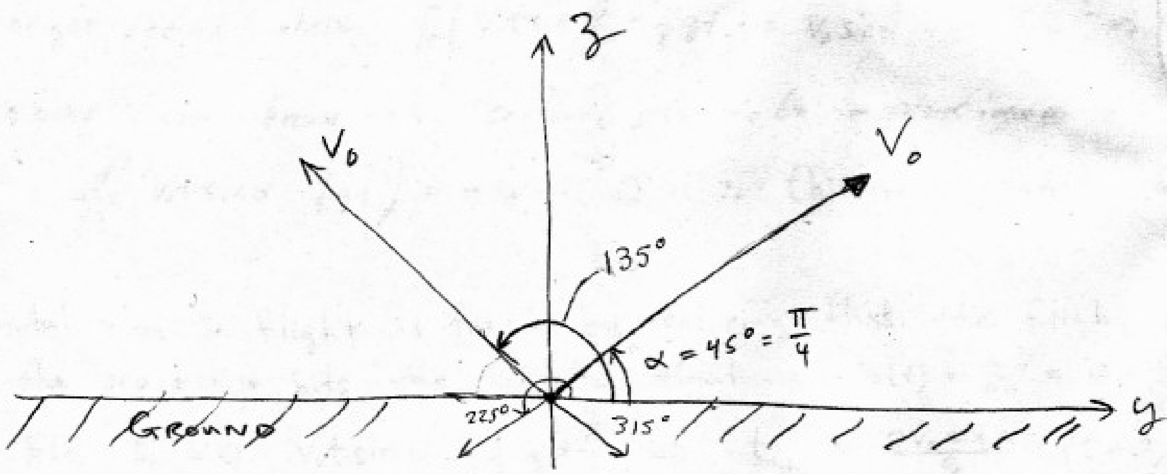
$$\frac{d^2R}{d\alpha^2} = \frac{2V_0^2}{g}\frac{d}{d\alpha}\left(\cos(2\alpha)\right) = \frac{-4V_0^2}{g}\left(\sin(2\alpha)\right)$$

Nice analysis

now $\cos(2\alpha) = 0$ for $\alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$

but for $\alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ $\frac{-4V_0^2}{g}\sin(2\alpha) < 0$ only for $\alpha = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$

$\frac{\pi}{4} \to 45°$, $\frac{5\pi}{4} \to 225°$, $\frac{9\pi}{4} \to 45°$, ... ← possible α for maximum range



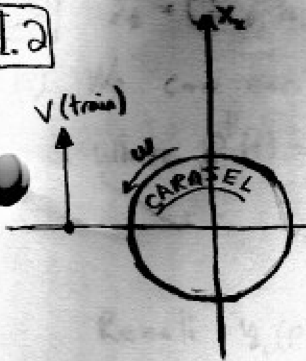
obviously the sol^{n} $\alpha = \frac{3\pi}{4}$ or $135° = \alpha$ corresponds to the "minimum" range as $\frac{d^2R}{d\alpha^2} = -\frac{4V_0^2}{g}\cancel{\sin\left(\frac{3\pi}{2}\right)}^{-1} > 0$

$\Rightarrow \alpha = 135°$ corresponds to a minimum. To conclude

$\boxed{\alpha = 45° \text{ and } \alpha = 135° \Rightarrow \text{maximum flight distance}}$

$\alpha = 225°$ and $\alpha = 315°$ are unphysical sol^{n}'s.

II.2



Let γ denote trajectory of train.
Let β denote trajectory of car.
GIVEN, using same notation as before,

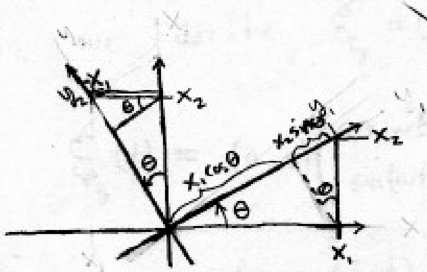
$$\gamma_x(0) = (-1,0) \longleftarrow$$ ~~Notation~~ OK my error

$$\dot{\gamma}_x(t) = v(0,1) = (0,v)$$

(a) $\dot{\gamma}_x(t) = \dfrac{d\gamma_x(t)}{dt} = (0,v) \Rightarrow \int_0^t d\gamma_x(\tilde{t}) = \gamma_x(t) - \gamma_x(0) = \int_0^t (0,v)\,d\tilde{t} = (0,v)t$

$$\therefore \boxed{\gamma_x(t) = (0,vt) + (-1,0)} = (-1, vt)$$

(b) $y_t(P) = X_t(P)\,\Phi^t(t) + \cancel{w(t)} \to 0$ as the "y" or carosel frame shares origin.



$$y_t(P) = X_t(P)\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix} \iff \begin{matrix} y_1 = x_1\cos\theta + x_2\sin\theta \\ y_2 = -x_1\sin\theta + x_2\cos\theta \end{matrix}$$

The transformation from $x \to y$ at a particular θ is clear from the geometry.

$$\omega \equiv \frac{d\theta}{dt} \Rightarrow \omega t = \theta(t) - \theta(0)$$

but $\theta(0) = 0$ as the carosel's y_1 axis is intially $\parallel$ to x_1 axis.
$\therefore\ \theta(t) = \omega t$.

$$\gamma_y(t) = \gamma_x(t)\,\Phi^t(t) = (-1, vt)\begin{pmatrix}\cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t\end{pmatrix}$$

$$\boxed{\gamma_y(t) = (-\cos\omega t + vt\sin\omega t,\ \sin\omega t + vt\cos\omega t)}$$

1.2

(c) Find velocity of train in both systems,

$$\dot{\gamma}_x(t) = \frac{d}{dt}\Big((0,vt)+(-1,0)\Big) = \boxed{(0,v) = \text{Velocity of train in } X}$$

$$\dot{\gamma}_y(t) = \frac{d}{dt}\Big(\big(vt\sin(\omega t) - \cos\omega t,\ \sin\omega t + vt\cos\omega t\big)\Big)$$

$$= \big(v\sin\omega t + v\omega t\cos\omega t + \omega\sin\omega t,\ \omega\cos\omega t + v\cos\omega t - vt\omega\sin\omega t\big)$$

$$\boxed{\big(\sin(\omega t)(v+\omega) + v\omega t\cos(\omega t),\ \cos(\omega t)(\omega+v) - v\omega t\sin\omega t\big) = \text{velocity of train in } y}$$

(d) At $t=0$ the car is about at $(-1,0)$ and has $\dot{\beta}_x(t) = \left(\frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}}\right)$

As always $\beta_x(t) - \beta_x(0) = \int_0^t \dot{\beta}_x(\tilde{t})\,d\tilde{t} = tu\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\therefore$ $\boxed{\beta_x(t) = tu\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) + (-1,0)}$

$$\beta_y(t) = \left(\frac{ut}{\sqrt{2}} - 1,\ \frac{ut}{\sqrt{2}}\right)\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix} = \left(\left(\frac{ut}{\sqrt{2}} - 1\right)\cos\theta + \frac{ut}{\sqrt{2}}\sin\theta,\ \left(1 - \frac{ut}{\sqrt{2}}\right)\sin\theta + \frac{ut}{\sqrt{2}}\cos\theta\right)$$

$$\boxed{\beta_y(t) = \left(\frac{ut}{\sqrt{2}}\big[\cos(\omega t) + \sin(\omega t)\big] - \cos(\omega t),\ \frac{ut}{\sqrt{2}}\big[\cos(\omega t) - \sin(\omega t)\big] - \sin(\omega t)\right)}$$

$\boxed{\text{I.2}}$ $z_t = (z_{t_1}, z_{t_2})$ oriented as X and locates it origin at train $(\gamma_x(t))$

(e) We can easily relate z to x coordinates by $z(P) = X_t(P)\,\varphi^t(t) - \gamma_x(t)$ where $\varphi^t(t) = I$ as they have same orientation and $-\gamma_x(t) = (+1, -vt)$ from before,

$\therefore\ z_t(P) = X_t(P) + (-1, vt)$

Recall $y_t(P) = X_t(P)\begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}$ thus $X_t(P) = y_t(P)\begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}$

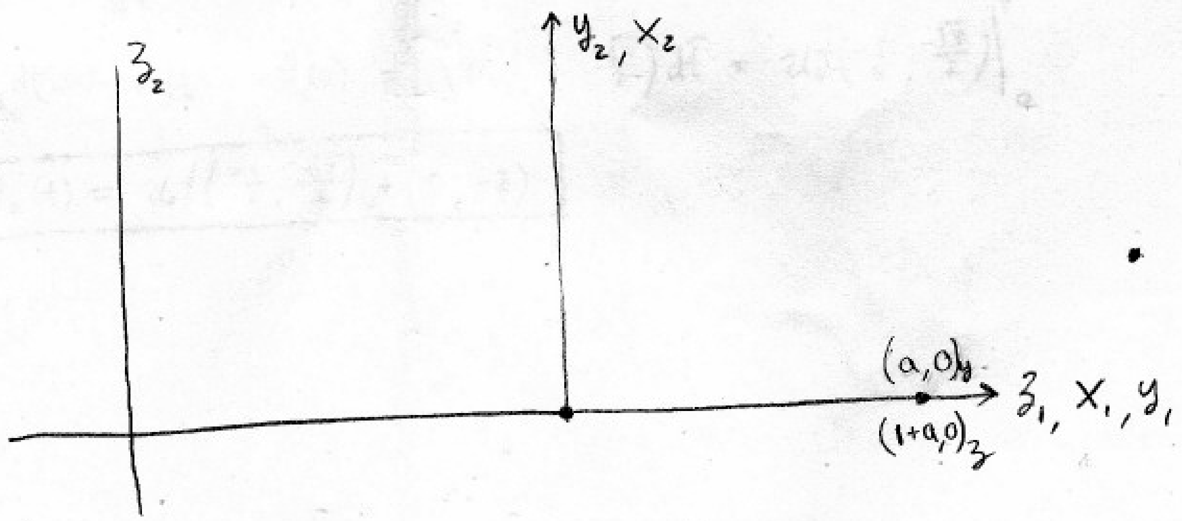
Now combine,

$$z_t(P) = X_t(P) + (-1, vt)$$
$$= y_t(P)\begin{pmatrix} \cos(\omega t) & \sin\omega t \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} + (-1, vt)$$

So we write $\xi_y = (a, 0)$ for point on rim of carosel, and find,

$$\xi_z(t) = (a,0)\begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} + (-1, vt)$$
$$= (a\cos(\omega t), a\sin(\omega t)) + (-1, vt)$$
$$\boxed{\xi_z(t) = (a\cos(\omega t) + 1,\ a\sin(\omega t) - vt)}$$

$t = 0$



$\xi_z(0) = (a\cos(0) + 1,\ 0) = (a, 0)$ it checks.