18. Given that $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$ and $x' = \gamma(x - \beta t)$, $x = \gamma(x' - \beta t')$, show that these equations can be solved to yield $t' = \gamma(t - \beta x)$.

- Since we need to solve for t', let's start by rearranging $x = \gamma(x' - \beta t')$ in terms of t'.

$$x = \gamma(x' - \beta t') \quad \rightarrow \quad t' = \frac{x}{\gamma\beta} - \frac{x'}{\beta}$$

- Now substitute the equation for x' into the equation for t'.

$$t' = \frac{x}{\gamma\beta} - \frac{x'}{\beta} \quad \rightarrow \quad t' = \frac{x}{\gamma\beta} - \frac{\gamma(x - \beta t)}{\beta} \quad \rightarrow \quad t' = \frac{x}{\gamma\beta} - \frac{\gamma x - \gamma\beta t}{\beta}$$

$$t' = \frac{x}{\gamma\beta} - \frac{\gamma x}{\beta} + \frac{\gamma\beta t}{\beta} \quad \rightarrow \quad t' = \frac{x}{\gamma\beta} - \frac{\gamma x}{\beta} + \gamma t \quad \rightarrow \quad t' = \gamma t - \left(\frac{\gamma}{\beta} - \frac{1}{\gamma\beta}\right)x$$

- Need to show that $\left(\dfrac{\gamma}{\beta} - \dfrac{1}{\gamma\beta}\right)$ simplifies down to $\gamma\beta$ in order to obtain $t' = \gamma(t - \beta x)$.

$$\frac{\gamma}{\beta} - \frac{1}{\gamma\beta} \quad \rightarrow \quad \frac{\gamma}{\gamma} \cdot \frac{\gamma}{\beta} - \frac{1}{\gamma\beta} \quad \rightarrow \quad \frac{\gamma^2}{\gamma\beta} - \frac{1}{\gamma\beta} \quad \rightarrow \quad \frac{\gamma^2 - 1}{\gamma\beta}$$

- Given that $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$,

$$\gamma^2 = \frac{1}{1-\beta^2} \quad \rightarrow \quad \gamma^2(1-\beta^2) = 1 \quad \rightarrow \quad \gamma^2 - \gamma^2\beta^2 = 1$$

- Substitute the new value for 1 into $\dfrac{\gamma^2 - 1}{\gamma\beta}$.

$$\frac{\gamma^2 - 1}{\gamma\beta} \quad \rightarrow \quad \frac{\gamma^2 - (\gamma^2 - \gamma^2\beta^2)}{\gamma\beta} \quad \rightarrow \quad \frac{\gamma^2 - \gamma^2 + \gamma^2\beta^2}{\gamma\beta} \quad \rightarrow \quad \frac{\gamma^2\beta^2}{\gamma\beta} \quad \rightarrow \quad \gamma\beta$$

- Thus, $\dfrac{\gamma}{\beta} - \dfrac{1}{\gamma\beta}$ can be simplified down to $\gamma\beta$. By substitution, we have

$$t' = \gamma t - \left(\frac{\gamma}{\beta} - \frac{1}{\gamma\beta}\right)x \quad \rightarrow \quad t' = \gamma t - \gamma\beta x \quad \rightarrow \quad t' = \gamma(t - \beta x).$$

19. Show that if the matrices below are the matrices of linear transformations on R^4 then they correspond to Lorentz transformations. In other words, show that $A^T \eta A = \eta$ for each matrix.

$$B_x = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_z = \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix}, \quad \tilde{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{bmatrix}$$

- $B_x^T \eta B_x$:

$$B_x^T \eta = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\gamma & -\gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left(B_x^T \eta\right) B_x = \begin{bmatrix} -\gamma & -\gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\gamma^2 + \gamma^2\beta^2 & \gamma^2\beta - \gamma^2\beta & 0 & 0 \\ \gamma^2\beta - \gamma^2\beta & -\gamma^2\beta^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$, then $\gamma^2 = \dfrac{1}{1-\beta^2} \rightarrow \gamma^2\left(1-\beta^2\right) = 1 \rightarrow \gamma^2 - \gamma^2\beta^2 = 1$

and thus $-\gamma^2 + \gamma^2\beta^2 = -1$. Therefore,

$$B_x^T \eta B_x = \begin{bmatrix} -\gamma^2 + \gamma^2\beta^2 & \gamma^2\beta - \gamma^2\beta & 0 & 0 \\ \gamma^2\beta - \gamma^2\beta & -\gamma^2\beta^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \eta$$

→ Thus, $B_x^T \eta B_x = \eta$.

- $B_z^T \eta B_z$:

$$B_z^T \eta = \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix}$$

$$\left(B_z^T \eta\right) B_z = \begin{bmatrix} -\gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix} \cdot \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix} = \begin{bmatrix} -\gamma^2+\gamma^2\beta^2 & 0 & 0 & \gamma^2\beta-\gamma^2\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma^2\beta-\gamma^2\beta & 0 & 0 & -\gamma^2\beta+\gamma^2 \end{bmatrix}$$

$$B_z^T \eta B_z = \begin{bmatrix} -\gamma^2+\gamma^2\beta^2 & 0 & 0 & \gamma^2\beta-\gamma^2\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma^2\beta-\gamma^2\beta & 0 & 0 & -\gamma^2\beta+\gamma^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \eta$$

→ Thus, $B_z^T \eta B_z = \eta$

- $\widetilde{R}^T \eta \widetilde{R}$:

$$\widetilde{R}^T \eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R^T & \\ 0 & & & \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R^T I & \\ 0 & & & \end{bmatrix}$$

$$\left(\widetilde{R}^T \eta\right) \widetilde{R} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R^T I & \\ 0 & & & \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R^T I R & \\ 0 & & & \end{bmatrix}$$

Since $R \in O(3)$, then $R^T R = I$, so $R^T I R = I$, which gives the following matrix:

$$\widetilde{R}^T \eta \widetilde{R} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R^T I R & \\ 0 & & & \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \eta$$

→ Thus, $\widetilde{R}^T \eta \widetilde{R} = \eta$

22. Consider a rod which is measured to have length L_o when at rest. Put the rod in motion such that it resides at the origin of the $\bar{\chi}$ which makes $\bar{\chi}$ the x-boosted frame w.r.t. x. Now measure the length of the rod w.r.t. x. We assume

$$\bar{\chi}(A) = (\bar{t}_A, 0) = (\bar{t}_A, \bar{x}_A) \text{ and } \bar{\chi}(B) = (\bar{t}_B, L_o) = (\bar{t}_B, \bar{x}_B)$$

When measuring the length we find $\chi^1(A)$ and $\chi^1(B)$ at the same time t, this means that

$$t = \gamma(\bar{t}_A + \beta\bar{x}_A) = \gamma(\bar{t}_B + \beta\bar{x}_B) \qquad \bar{t}_A = \bar{t}_B + \beta L_o$$

Note $\chi^1(A) = \gamma(\bar{x}_A + \beta\bar{t}_A)$ and $\chi^1(B) = \gamma(\bar{x}_B + \beta\bar{t}_B)$. Show $\chi^1(B) - \chi^1(A) = \dfrac{L_o}{\gamma}$.

$$\begin{aligned}
\chi^1(B) - \chi^1(A) &= \gamma(\bar{x}_B + \beta\bar{t}_B) - \gamma(\bar{x}_A + \beta\bar{t}_A) \\
&= \gamma\bar{x}_B + \gamma\beta\bar{t}_B - \gamma\bar{x}_A - \gamma\beta\bar{t}_A \\
&= \gamma\bar{x}_B + \gamma\beta\bar{t}_B - \gamma\bar{x}_A - \gamma\beta(\bar{t}_B + \beta L_o) && \text{Substitute for } \bar{t}_A \\
&= \gamma\bar{x}_B + \gamma\beta\bar{t}_B - \gamma\bar{x}_A - \gamma\beta\bar{t}_B - \gamma\beta^2 L_o \\
&= \gamma\bar{x}_B - \gamma\bar{x}_A - \gamma\beta^2 L_o && \text{Cancel out the } \gamma\beta\bar{t}_B \text{ terms} \\
&= \gamma(\bar{x}_B - \bar{x}_A) - \gamma\beta^2 L_o \\
&= \gamma(L_o - 0) - \gamma\beta^2 L_o && \text{Since } \bar{x}_B = L_o \;\&\; \bar{x}_A = 0 \\
&= \gamma L_o - \gamma\beta^2 L_o \\
&= \gamma L_o(1 - \beta^2) \\
&= \gamma L_o\left(\frac{1}{\gamma^2}\right) && \gamma = \frac{1}{\sqrt{1-\beta^2}} \to \gamma^2 = \frac{1}{1-\beta^2} \to \frac{1}{\gamma^2} = 1 - \beta^2 \\
&= \frac{L_o}{\gamma}
\end{aligned}$$

23. Consider a fixed point in the moving $\bar{\chi}$-frame as in problem 22. Suppose a clock at $\bar{x} = \bar{x}_0$ ticks from $\bar{t}_1$ to time $\bar{t}_2$. We label these two "events" by $\bar{\chi}(A) = (\bar{t}_1, \bar{x}_o)$ and $\bar{\chi}(B) = (\bar{t}_2, \bar{x}_o)$.

Show that $\chi^o(B) - \chi^o(A) = \gamma(\bar{t}_2 - \bar{t}_1)$.
Since time is varying, we have $\chi^o(A) = \gamma(\bar{t}_1 + \beta\bar{x}_o)$ and $\chi^o(B) = \gamma(\bar{t}_2 + \beta\bar{x}_o)$

$$\begin{aligned}
\chi^o(B) - \chi^o(A) &= \gamma(\bar{t}_2 + \beta\bar{x}_o) - \gamma(\bar{t}_1 + \beta\bar{x}_o) \\
&= \gamma\bar{t}_2 + \gamma\beta\bar{x}_o - \gamma\bar{t}_1 - \gamma\beta\bar{x}_o \\
&= \gamma\bar{t}_2 - \gamma\bar{t}_1 && \text{Cancel out the } \gamma\beta\bar{x}_o \text{ terms} \\
&= \gamma(\bar{t}_2 - \bar{t}_1)
\end{aligned}$$

24. Find a *counter-example* to the triangle-inequality for the Minkowski metric. In particular, find $A,B,C \in R^4$ such that $\overline{AB} = \left(B^0 - A^0, B^1 - A^1, B^2 - A^2, B^3 - A^3\right)$ and $\left\langle \overline{AB}, \overline{AB} \right\rangle + \left\langle \overline{BC}, \overline{BC} \right\rangle \leq \left\langle \overline{AC}, \overline{AC} \right\rangle$.

Let $A = (0,0,0,0)$, $B = (1,2,0,0)$, and $C = (3,5,0,0)$

Then, $\overline{AB} = (1-0, 2-0, 0-0, 0-0) = (1,2,0,0)$
$\overline{BC} = (3-1, 5-2, 0-0, 0-0) = (2,3,0,0)$
$\overline{AC} = (3-0, 5-0, 0-0, 0-0) = (3,5,0,0)$

$$\left\langle \overline{AB}, \overline{AB} \right\rangle = -\overline{AB}^0 \cdot \overline{AB}^0 + \overline{AB}^1 \cdot \overline{AB}^1 + \overline{AB}^2 \cdot \overline{AB}^2 + \overline{AB}^3 \cdot \overline{AB}^3 = -1^2 + 2^2 + 0^2 + 0^2 = -1 + 4 = 3$$
$$\left\langle \overline{BC}, \overline{BC} \right\rangle = -\overline{BC}^0 \cdot \overline{BC}^0 + \overline{BC}^1 \cdot \overline{BC}^1 + \overline{BC}^2 \cdot \overline{BC}^2 + \overline{BC}^3 \cdot \overline{BC}^3 = -2^2 + 3^2 + 0^2 + 0^2 = -4 + 9 = 5$$
$$\left\langle \overline{AC}, \overline{AC} \right\rangle = -\overline{AC}^0 \cdot \overline{AC}^0 + \overline{AC}^1 \cdot \overline{AC}^1 + \overline{AC}^2 \cdot \overline{AC}^2 + \overline{AC}^3 \cdot \overline{AC}^3 = -3^2 + 5^2 + 0^2 + 0^2 = -9 + 25 = 16$$

Since $3 + 5 \leq 16$, we have found a counter-example to the triangle inequality such that
$$\left\langle \overline{AB}, \overline{AB} \right\rangle + \left\langle \overline{BC}, \overline{BC} \right\rangle \leq \left\langle \overline{AC}, \overline{AC} \right\rangle.$$

(correct, but my problem is to easy instead should look at $\sqrt{K}$, >1 .)

25. If we have two events $A = (t_A, x_A, y_A, z_A)$ and $B = (t_B, x_B, y_B, z_B)$ which are separated by a timelike vector $(\langle B-A, B-A \rangle < 0)$ then show $t_A < t_B \Rightarrow \bar{t}_A < \bar{t}_B$ where $\bar{t}$ is the time in the x-boosted frame relative to the frame where we have described *A* and *B*, $\bar{t} = \gamma(t - \beta x)$, $\bar{x} = \gamma(x - \beta t)$, $\bar{y} = y$, $\bar{z} = z$.

Given $\bar{t} = \gamma(t - \beta x)$, then $\bar{t}_A = \gamma(t_A - \beta x_A)$ and $\bar{t}_B = \gamma(t_B - \beta x_B)$. We let $A = (t_A, 0, 0, 0)$ and $B = (t_B, 0, 0, 0)$ which means that our equations become $\bar{t}_A = \gamma t_A$ and $\bar{t}_B = \gamma t_B$.

We start with $t_A < t_B$.

Since $\gamma > 0$, then we can multiply both sides of the inequality by γ without changing it.

$t_A < t_B \rightarrow \gamma t_A < \gamma t_B$

From above, we found that $\bar{t}_A = \gamma t_A$ and $\bar{t}_B = \gamma t_B$. Thus, by substitution,

$\gamma t_A < \gamma t_B \rightarrow \bar{t}_A < \bar{t}_B$

Therefore, $t_A < t_B \Rightarrow \bar{t}_A < \bar{t}_B$.

✓

26. Prove **proposition** 7.10.6: The relativistic force, $\vec{F} = \frac{1}{c^2}(\vec{F}\cdot\vec{u})\vec{u} + m\frac{d\vec{u}}{dt}$, reduces as follows in the **special case** that (1.) the $\vec{a}, \vec{F}, \vec{u}$ are parallel, and (2.) the force is perpendicular to $\vec{u}$, *a.k.a.* $\vec{F}\cdot\vec{u} = 0$.

(1.) $\vec{F}_{|} = m_o\gamma^3(u)\frac{d\vec{u}}{dt}$ (2.) $\vec{F}_{\perp} = m_o\gamma(u)\frac{d\vec{u}}{dt}$

- (1.) If $\vec{a}, \vec{F}, \vec{u}$ are parallel, then $(\vec{F}\cdot\vec{u}) = \vec{F}\vec{u}$ and: (need to drop vectors throughout, just look at magnitudes) $\vec{F}\cdot\vec{u} = Fu$.

$$\vec{F} = \frac{1}{c^2}\vec{F}\vec{u}\vec{u} + m\frac{d\vec{u}}{dt} \rightarrow \vec{F} = \frac{1}{c^2}\vec{F}\vec{u}^2 + m\frac{d\vec{u}}{dt} \rightarrow \vec{F} = \vec{F}\frac{\vec{u}^2}{c^2} + m\frac{d\vec{u}}{dt}$$

$$\vec{F} - \vec{F}\frac{\vec{u}^2}{c^2} = m\frac{d\vec{u}}{dt} \rightarrow \vec{F}\left(1 - \frac{\vec{u}^2}{c^2}\right) = m\frac{d\vec{u}}{dt} \rightarrow \vec{F}\left(\frac{1}{\gamma^2(u)}\right) = m\frac{d\vec{u}}{dt}$$

$$\vec{F} = \gamma^2(u)\cdot m\frac{d\vec{u}}{dt} \rightarrow \vec{F} = \gamma^2(u)\cdot m_o\gamma(u)\frac{d\vec{u}}{dt} \rightarrow \vec{F}_{\|} = m_o\gamma^3(u)\frac{d\vec{u}}{dt}$$

- (2.) If the force is perpendicular to $\vec{u}$, *a.k.a.* $\vec{F}\cdot\vec{u} = 0$, then:

$$\vec{F}_{\perp} = \frac{1}{c^2}(0)\,\vec{u} + m\frac{d\vec{u}}{dt} = m\frac{d\vec{u}}{dt}$$

By definition 7.10.1, $m = m_o\gamma(u)$. Thus, by substitution, ✓

$$\vec{F}_{\perp} = m_o\gamma(u)\frac{d\vec{u}}{dt}$$

27. Prove proposition 7.10.7 –
Relativistic Charged Particle in Circular Motion: Let us suppose we have a charge q moving with initial velocity $(0, -v_0, 0)$ subject to the constant magnetic field $\vec{B} = (0, 0, B)$. Everything is the same as before except now let the initial position be $(mv_0 / qB, 0, 0)$ where $m = \gamma(v_0)m_0$ is the relativistic mass. It can be shown that the particle travels in a circle of radius $R = \gamma(v_0)m_0 v_0 / qB$ centered at the origin lying in the $z = 0$ plane.

Start with $F = q\vec{u} \times \vec{B} = m\frac{d\vec{u}}{dt} = m_0\gamma\frac{d\vec{u}}{dt}$

$$m_0\gamma\frac{d\vec{r}}{dt} = q\left(\vec{v} \times \vec{B}\right)$$

$$m_0\gamma\left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right) = qB\left(\frac{dy}{dt}, -\frac{dx}{dt}, 0\right)$$

$$\left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right) = \frac{qB}{m_0\gamma}\left(\frac{dy}{dt}, -\frac{dx}{dt}, 0\right)$$

$$\left.\begin{array}{l} \dfrac{d^2x}{dt^2} = \alpha\dfrac{dy}{dt} \\ \dfrac{d^2y}{dt^2} = -\alpha\dfrac{dx}{dt} \\ \dfrac{d^2z}{dt^2} = 0 \end{array}\right\} \text{ where } \alpha = \frac{qB}{m_0\gamma}$$

Notice that $\frac{dy}{dt} = \frac{1}{\alpha}\frac{d^2x}{dt^2} \Rightarrow \frac{d^2y}{dt^2} = \frac{1}{\alpha}\frac{d^3x}{dt^3} = -\alpha\frac{dx}{dt} \Rightarrow \frac{d^3x}{dt^3} = -\alpha^2\frac{dx}{dt}$

Thus, we need to solve $\frac{d^3x}{dt^3} = -\alpha^2\frac{dx}{dt}$, so we introduce $w = \frac{dx}{dt}$.

Our equation then becomes $\frac{d^2w}{dt^2} = -\alpha^2 w$, which has the known solution of

$$w = \frac{dx}{dt} = c_1\cos(\alpha t) + c_2\sin(\alpha t)$$

We can now calculate everything else from this equation.

$$\frac{dy}{dt} = \frac{1}{\alpha}\frac{d^2x}{dt^2} = \frac{1}{\alpha}\left(-c_1\alpha\sin(\alpha t) + c_2\alpha\cos(\alpha t)\right) = -c_1\sin(\alpha t) + c_2\cos(\alpha t)$$

We now integrate both ***dx/dt* and *dy/dt*** equations to find $x(t)$ and $y(t)$.

$$x(t) = \frac{1}{\alpha}\left(c_1\sin(\alpha t) - c_2\cos(\alpha t)\right) + c_3$$

$$y(t) = \frac{1}{\alpha}\left(c_1\cos(\alpha t) + c_2\sin(\alpha t)\right) + c_4$$

Finally, apply the initial conditions:

$$\frac{dx}{dt}(0) = 0 = c_1 \quad \Rightarrow c_1 = 0$$

$$\frac{dy}{dt}(0) = -v_0 = c_2 \quad \Rightarrow c_2 = -v_0$$

$$x(0) = \frac{m_0\gamma v_0}{qB} = \frac{v_0}{\alpha} = \frac{1}{\alpha}(v_0) + c_3 \quad \Rightarrow c_3 = 0$$

$$y(0) = 0 = \frac{1}{\alpha}(0) + c_4 \quad \Rightarrow c_4 = 0$$

Therefore, since $\frac{c_2}{\alpha} = \frac{-v_0}{\alpha} = \frac{-m_0\gamma v_0}{qB}$, then

$$x(t) = \left(\frac{m_0\gamma v_0}{qB}\right)\cos\left(\frac{qBt}{m_0\gamma}\right)$$

$$y(t) = -\left(\frac{m_0\gamma v_0}{qB}\right)\sin\left(\frac{qBt}{m_0\gamma}\right)$$

$$z(t) = 0$$

This is the circle $x^2 + y^2 = R^2$ centered at the origin with radius $R = \left(\frac{m_0\gamma(v_0)v_0}{qB}\right)$, lying in the $z = 0$ plane.