28. Let V be the vector space with $\{e_i\}_{i=1}^n$ an ordered basis. Define the dual-space to V $V^* \equiv \{\alpha : V \to R \mid \alpha \text{ a linear mapping}\}$. Define dual basis $\{e^1, e^2, \ldots, e^n\} = \{e^j\}_{j=1}^n$ by $e^j(e_i) = \delta_i^j$ where we assume $e^j : V \to R$ are linear maps. We give V^* the structure of a vector space as follows

$$(\alpha+\beta)(x) = \alpha(x) + \beta(x) \qquad \forall \alpha, \beta \in V^* \text{ and } \forall c \in R \text{ and } x \in V$$
$$(c\alpha)(x) = c\alpha(x)$$

Show,

(i) V^* is closed under vector addition and scalar multiplication

$$\begin{aligned}(\alpha+\beta)(x+cy) &= \alpha(x+cy) + \beta(x+cy) \\ &= \alpha(x) + \alpha(cy) + \beta(x) + \beta(cy) \\ &= \alpha(x) + \beta(x) + c\alpha(y) + c\beta(y) \\ &= \alpha(x) + \beta(x) + c(\alpha(y) + \beta(y)) \\ &= (\alpha+\beta)(x) + c(\alpha+\beta)(y)\end{aligned}$$

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$$\begin{aligned}(b\alpha)(x+cy) &= \cancel{(b\alpha)(x) + (b\alpha)(cy)} \\ &= \cancel{(b\alpha)(x) + c(b\alpha)(y)}\end{aligned}$$

$$\begin{aligned} b(\alpha(x+cy)) &= b[\alpha(x) + c\alpha(y)] \\ &= b\alpha(x) + c\, b\alpha(y) \\ &= (b\alpha)(x) + c\,(b\alpha)(y). \end{aligned}$$

(ii) $\{e^j\}_{j=1}^n$ forms a basis for V^*

Need to show linear independence:

Assume $c_j e^j = 0$, then

$(c_j e^j)(e_k) = 0(e_k) = 0$ where e_k is the kth basis element of V

$c_j \delta_k^j = 0$ since $e^j(e_k) = \delta_k^j$

$c_k = 0$

Need to show spanning:

Claim $\alpha = \alpha_i e^i$, then

$\alpha(e_j) = \alpha_i e^i(e_j)$

By definition, $\alpha(e_j) = \alpha_j$ and $e^i(e_j) = \delta_j^i$, so

$\alpha_j = \alpha_i \delta_j^i = \alpha_j$

thus $\alpha = \alpha_i e^i$ since we just demonstrated the maps agree on the basis and that's sufficient as α and $\alpha_i e^i$ are linear operators.

(iii) If $V = R^n$ and $\{e_i\}$ are column vectors then for each $\alpha \in V^* \; \exists \; a \in R^n$ such that

$$\alpha(x) = a^T x \quad \forall \; x \in R^n$$

$$\begin{aligned}\alpha(x) &= (\alpha_i e^i)(x) \\ &= (\alpha_i e^i)(x^j e_j) \\ &= \alpha_i x^j e^i(e_j) \\ &= \alpha_i x^j \delta_i^j \\ &= \alpha_j x^j\end{aligned}$$

Since $\alpha(x) = \alpha_j x^j$, then $\alpha_j x^j = a^T x = (a_1, a_2, \ldots, a_n)\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}$.

Therefore, $a^T = (\alpha_j)$. So $a = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$.

29. Let $B = \{b : V \times V \to R \mid b \text{ is bilinear}\}$.

(a) Show that B forms a vector space with respect to operations,

$$\begin{aligned}(b+m)(x,y) &= b(x,y) + m(x,y) \\ (cb)(x,y) &= cb(x,y)\end{aligned} \qquad \forall \; m, b \in B \text{ and } \forall \; x, y \in V \text{ and } c \in R$$

$$\begin{aligned}(b+m)(x+cy,z) &= b(x+cy,z) + m(x+cy,z) \\ &= b(x,z) + b(cy,z) + m(x,z) + m(cy,z) \\ &= b(x,z) + m(x,z) + cb(y,z) + cm(y,z) \\ &= b(x,z) + m(x,z) + c(b(y,z) + m(y,z)) \\ &= (b+m)(x,z) + c(b+m)(y,z)\end{aligned}$$

$$\begin{aligned}(b+m)(x,y+cz) &= b(x,y+cz) + m(x,y+cz) \\ &= b(x,y) + b(x,cz) + m(x,y) + m(x,cz) \\ &= b(x,y) + m(x,y) + cb(x,z) + cm(x,z) \\ &= b(x,y) + m(x,y) + c(b(x,z) + m(x,z)) \\ &= (b+m)(x,y) + c(b+m)(x,z)\end{aligned}$$

assumes what we're to prove.

$$(cb)(x+dy,z) = \cancel{cb(x,z)+cb(dy,z)}$$
$$= \cancel{cb(x,z)+dcb(y,z)}$$

$$(cb)(x,y+dz) = \cancel{cb(x,y)+cb(x,dz)}$$
$$= \cancel{cb(x,y)+dcb(x,z)}$$

$$cb(x+dy,z) = c[b(x+dy,z)]$$
$$= c[b(x,z)+d\,b(y,z)]$$
$$= cb(x,z) + dcb(y,z)$$
$$= (cb)(x,z) + d(cb)(y,z).$$

and a similar calculation for the proof of

$$(cb)(x,y+dz) = (cb)(x,y) + d(cb)(x,z).$$

(b) Show that $\{e^i \otimes e^j\}_{i,j=1}^n$ forms a basis for B.

(i) Need to show linear independence.
Assume $c_{ij}e^i \otimes e^j = 0$, then

$$c_{ij}e^i \otimes e^j(e_k,e_l) = 0(e_k,e_l) = 0$$
$$c_{ij}e^i(e_k)e^j(e_l) = 0$$
$$c_{ij}\delta^i_k\delta^j_l = 0$$
$$c_{kl} = 0$$

(ii) Need to show spanning. $b \in B$ *then* $\exists\, b_{ij}$ *s.t.* $b = b_{ij}e^i \otimes e^j$.

Claim $b = b_{ij}e^i \otimes e^j$, then

$$b(e_k,e_l) = b_{ij}e^i \otimes e^j(e_k,e_l)$$

$b_{kl} = b_{ij}e^i(e_k)e^j(e_l)$ since $b(e_k,e_l) = b_{kl}$ by defn.

$b_{kl} = b_{ij}\delta^i_k\delta^j_l$ since $e^i(e_k)e^j(e_l) = \delta^i_k\delta^j_l$

$b_{kl} = b_{kl}$ $\quad\therefore b = b_{ij}e^i \otimes e^j$ since we just checked they agree on the basis.

(iii) Show that $e^i \otimes e^j \in B\ \forall\, i,j \in \{1,2,\ldots,n\}$. (Need to show bilinear)

$$\begin{aligned}(e^i \otimes e^j)(x+cy,z) &= e^i(x+cy)e^j(z)\\ &= [e^i(x)+e^i(cy)]e^j(z)\\ &= e^i(x)e^j(z)+ce^i(y)e^j(z)\\ &= (e^i \otimes e^j)(x,z)+c(e^i \otimes e^j)(y,z)\end{aligned}$$

$$\begin{aligned}(e^i \otimes e^j)(x,y+cz) &= e^i(x)e^j(y+cz)\\ &= e^i(x)[e^j(y)+e^j(cz)]\\ &= e^i(x)e^j(y)+ce^i(x)e^j(z)\\ &= (e^i \otimes e^j)(x,y)+c(e^i \otimes e^j)(x,z)\end{aligned}$$

MA 430 Homework, contd.

30 1. Let $h : \mathbb{R}^3 \times \mathbb{R}^3 \times R^3 \to \mathbb{R}$, and $h(u,v,w) = u \cdot (v \times w), u, v, w \in \mathbb{R}^3$. Show it is an antisymmetric trilinear mapping.

For antisymmetry, need to show that

$$h(u,v,w) = h(w,u,v) = h(v,w,u) = -h(v,u,w) = -h(u,w,v) = -h(w,v,u)$$

Luckily, we know that

$$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$$

and so $h(u,v,w)$ is

$$\underbrace{v \cdot (w \times u)}_{h(v,w,u)} = \overbrace{-v \cdot (u \times w)}^{-h(v,u,w)} = \underbrace{-u \cdot (w \times v)}_{-h(u,w,v)}$$

which continues to equal

$$\underbrace{w \cdot (u \times v)}_{h(w,u,v)} = \overbrace{-w \cdot (v \times u)}^{-h(w,v,u)} = \underbrace{u \cdot (v \times w)}_{h(u,v,w)}$$

Which is *clearly* anti-symmetric. To show linearity, it needs to be showin each slot. Let $c \in \mathbb{R}, u, v, w, a \in \mathbb{R}^3$ For the first slot:

$$\begin{aligned} h(u+ca,v,w) &= (u+ca) \cdot (v \times w) \\ &= u \cdot (v \times w) + ca(v \times w) \\ &= h(u,v,w) + ch(a,v,w) \end{aligned}$$

Second slot:

$$\begin{aligned} h(u,v+ca,w) &= u \cdot ((v+ca) \times w) \\ &= u \cdot (v \times w + ca \times w) \\ &= u \cdot (v \times w) + u \cdot (ca \times w) \\ &= h(u,v,w) + ch(u,a,w) \end{aligned}$$

Third slot:

$$\begin{aligned} h(u,v,w+ca) &= u \cdot (v \times (w+ca)) \\ &= u \cdot (v \times w + v \times ca) \\ &= u \cdot (v \times w) + u \cdot (v \times ca) \\ &= h(u,v,w) + ch(u,v,a) \end{aligned}$$

And we have linearity.

3. 2. Show that $\{e_i \otimes e_j\}_{i,j=1}^n$ forms a basis of all bilinear forms on V^*. Need to show that $\{e_i \otimes e_j\}_{i,j=1}^n$ spans, and is linearly independent on V^*. Also need to show bilinearity.

Linear Independence: Let $C^{ij} \in \mathbb{R}$, and $e_i, e_j \in V^*, e^k, e^l \in V$. If it is linearly independent, $C^{ij}(e_i \otimes e_j) = 0 \iff C^{ij} = 0$. So,

$$\begin{aligned} C^{ij}(e_i \otimes e_j) &= 0 \\ C^{ij}(e_i \otimes e_j)(e^k, e^l) &= 0\ (e^k, e^l) \\ C^{ij} e_i(e^k) e_j(e^l) &= 0 \\ C^{ij} \delta_i^k \delta_j^l &= 0 \\ C^{kl} &= 0 \end{aligned}$$

rather $\underline{e^k(e_i) e^l(e_j)} = \delta_i^k \delta_j^l$

Therefore, $\{e_i \otimes e_j\}_{i,j=1}^n$ is linearly independent. For spanning, we claim that $b \in V^*$ can be written as $b = b^{ij}(e_i \otimes e_j)$. Then,

$$\begin{aligned} b(e^k, e^l) &= b^{ij}(e_i \otimes e_j)(e^k, e^l) \\ b^{kl} &= b^{ij} \cancel{e_i(e^k) e_j(e^l)} \\ &= b^{ij}\ \delta_i^k \delta_j^l \\ b^{kl} &= b^{kl} \end{aligned}$$

again by def^{n}:
$(e_i \otimes e_j)(e^k, e^l) = e^k(e_i)\, e^l(e_j)$.

And so it spans. Lastly, bilinearity. $c \in \mathbb{R}, \alpha, \beta, \gamma \in V$.

$$\begin{aligned} (e_i \otimes e_j)(\alpha + c\beta, \gamma)) &= (\alpha + c\beta)(e_i)\gamma(e_j) \\ &= (\alpha(e_i) + c\beta(e_i))\gamma(e_j) \\ &= \alpha(e_i)\gamma(e_j) + c\beta(e_i)\gamma(e_j) \\ &= (e_i \otimes e_j)(\alpha, \gamma) + c(e_i \otimes e_j)(\beta, \gamma) \end{aligned}$$

Need to also show bilinearity in the second "slot".

$$\begin{aligned} (e_i \otimes e_j)(\alpha, \beta + c\gamma) &= \alpha(e_i)(\beta + c\gamma)(e_j) \\ &= \alpha(e_i)(\beta(e_j) + c\gamma(e_j)) \\ &= \alpha(e_i)\beta(e_j) + c\alpha(e_i)\gamma(e_j) \\ &= (e_i \otimes e_j)(\alpha, \beta) + c(e_i \otimes e_j)(\alpha, \gamma) \end{aligned}$$

And so after all of these steps, $\{e_i \otimes e_j\}_{i,j=1}^n$ forms a basis of all bilinear forms on V^*.