32. Verify the claims of Example 9.5.8. That is show

$$T_i^{\ jk} \equiv g_{il} T^{ljk} = (T')_i^{\ jk}$$
$$T_{ij}^{\ \ k} \equiv g_{il} g_{jm} T^{lmk} = (T'')_{ij}^{\ \ k}$$
$$T_{ijk} \equiv g_{il} g_{jm} g_{kn} T^{lmn} = (T''')_{ijk}$$

$$\begin{aligned}
(T')_i^{\ jk} &= T'(e_i, e^j, e^k) \\
&= T(g(e_i, \bullet), e^j, e^k) \\
&= T(\alpha_{e_i}, e^j, e^k) \\
&= T(g_{il} e^l, e^j, e^k) && \text{since } \alpha_{e_i} = g_{il} e^l \\
&= g_{il} T(e^l, e^j, e^k) \\
&= g_{il} T^{ljk} \\
&= T_i^{\ jk}
\end{aligned}$$

$$\begin{aligned}
(T'')_{ij}^{\ \ k} &= T''(e_i, e_j, e^k) \\
&= T(g(e_i, \bullet), g(e_j, \bullet), e^k) \\
&= T(\alpha_{e_i}, \alpha_{e_j}, e^k) \\
&= T(g_{il} e^l, g_{jm} e^m, e^k) \\
&= g_{il} g_{jm} T(e^l, e^m, e^k) \\
&= g_{il} g_{jm} T^{lmk} \\
&= T_{ij}^{\ \ k}
\end{aligned}$$

$$\begin{aligned}
(T''')_{ijk} &= T'''(e_i, e_j, e_k) \\
&= T(g(e_i, \bullet), g(e_j, \bullet), g(e_k, \bullet)) \\
&= T(\alpha_{e_i}, \alpha_{e_j}, \alpha_{e_k}) \\
&= T(g_{il} e^l, g_{jm} e^m, g_{kn} e^n) \\
&= g_{il} g_{jm} g_{kn} T(e^l, e^m, e^n) \\
&= g_{il} g_{jm} g_{kn} T^{lmn} \\
&= T_{ijk}
\end{aligned}$$

✓

33. Suppose we have a tensor $G = G_{\mu\nu} e^{\mu} \otimes e^{\nu}$ in Minkowski space where

$$(G_{\mu\nu}) = \begin{bmatrix} a & b & c & d \\ 0 & 1 & 2 & 3 \\ x & 0 & 0 & 0 \\ y & z & 0 & 0 \end{bmatrix}$$

where $a,b,c,d,x,y,z \in R$. Find $(G^{\mu\nu})$ and present your answer as a matrix.

$$G^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}G_{\alpha\beta} = \eta^{\mu\alpha}G_{\alpha\beta}\eta^{\beta\nu} = (\eta G \eta)^{\mu\nu}$$

$$G^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b & c & d \\ 0 & 1 & 2 & 3 \\ x & 0 & 0 & 0 \\ y & z & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -a & -b & -c & -d \\ 0 & 1 & 2 & 3 \\ x & 0 & 0 & 0 \\ y & z & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a & -b & -c & -d \\ 0 & 1 & 2 & 3 \\ -x & 0 & 0 & 0 \\ -y & z & 0 & 0 \end{bmatrix}$$

34. Let $F = F_{\mu\nu} e^{\mu} \otimes e^{\nu}$ be the field tensor defined in eqn. 9.15 with respect to the coordinate system with basis $\{e_\mu\}$ and dual basis $\{e^\nu\}$. Suppose $\{\bar{e}_\mu\}$ is another coordinate system related to the original system as

$$\bar{e}_\mu = (\Lambda^{-1})^{\nu}_{\ \mu} e_\nu \quad \text{and} \quad \bar{e}^\mu = \Lambda^{\mu}_{\ \nu} e^\nu$$

denote the components of F with respect to new coordinates by $\bar{F}_{\mu\nu} = F(\bar{e}_\mu, \bar{e}_\nu)$. Show that $F_{\mu\nu} F^{\mu\nu} = \bar{F}_{\mu\nu} \bar{F}^{\mu\nu}$.

$$\begin{aligned}
F &= F_{\mu\nu} e^{\mu} \otimes e^{\nu} \\
F(\bar{e}_\mu, \bar{e}_\nu) &= F_{\mu\nu} e^{\mu} \otimes e^{\nu}(\bar{e}_\mu, \bar{e}_\nu) \\
\bar{F}_{\mu\nu} &= F_{\mu\nu} e^{\mu}(\bar{e}_\mu) e^{\nu}(\bar{e}_\nu) \\
&= F_{\mu\nu} e^{\mu}\left((\Lambda^{-1})^{\alpha}_{\ \mu} e_\alpha\right) e^{\nu}\left((\Lambda^{-1})^{\beta}_{\ \nu} e_\beta\right) \\
&= (\Lambda^{-1})^{\alpha}_{\ \mu} (\Lambda^{-1})^{\beta}_{\ \nu} F_{\mu\nu} e^{\mu}(e_\alpha) e^{\nu}(e_\beta) \\
&= (\Lambda^{-1})^{\alpha}_{\ \mu} (\Lambda^{-1})^{\beta}_{\ \nu} F_{\mu\nu} \delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} \\
&= (\Lambda^{-1})^{\alpha}_{\ \mu} (\Lambda^{-1})^{\beta}_{\ \nu} F_{\alpha\beta}
\end{aligned}$$

$$\begin{aligned}
F &= F^{\mu\nu} e_{\mu} \otimes e_{\nu} \\
F(\bar{e}^\mu, \bar{e}^\nu) &= F^{\mu\nu} e_{\mu} \otimes e_{\nu}(\bar{e}^\mu, \bar{e}^\nu) \\
\bar{F}^{\mu\nu} &= F^{\mu\nu} e_{\mu}(\bar{e}^\mu) e_{\nu}(\bar{e}^\nu) \\
&= F^{\mu\nu} e_{\mu}(\Lambda^{\mu}_{\ \gamma} e^\gamma) e_{\nu}(\Lambda^{\nu}_{\ \sigma} e^\sigma) \\
&= \Lambda^{\mu}_{\ \gamma} \Lambda^{\nu}_{\ \sigma} F^{\mu\nu} e_{\mu}(e^\gamma) e_{\nu}(e^\sigma) \\
&= \Lambda^{\mu}_{\ \gamma} \Lambda^{\nu}_{\ \sigma} F^{\mu\nu} \delta^{\gamma}_{\mu} \delta^{\sigma}_{\nu} \\
&= \Lambda^{\mu}_{\ \gamma} \Lambda^{\nu}_{\ \sigma} F^{\gamma\sigma}
\end{aligned}$$

$$\begin{aligned}
\bar{F}_{\mu\nu} \bar{F}^{\mu\nu} &= (\Lambda^{-1})^{\alpha}_{\ \mu} (\Lambda^{-1})^{\beta}_{\ \nu} F_{\alpha\beta} \Lambda^{\mu}_{\ \gamma} \Lambda^{\nu}_{\ \sigma} F^{\gamma\sigma} \\
&= \Lambda^{\mu}_{\ \gamma} (\Lambda^{-1})^{\alpha}_{\ \mu} \Lambda^{\nu}_{\ \sigma} (\Lambda^{-1})^{\beta}_{\ \nu} F_{\alpha\beta} F^{\gamma\sigma} \\
&= \delta^{\alpha}_{\gamma} \delta^{\beta}_{\sigma} F_{\alpha\beta} F^{\gamma\sigma} \\
&= F_{\alpha\beta} F^{\alpha\beta}
\end{aligned}$$

$$\therefore \ \bar{F}_{\mu\nu} \bar{F}^{\mu\nu} = F_{\mu\nu} F^{\mu\nu}$$

35. Consider an x-boost,

$$\left(\Lambda^{\mu}_{\ \nu}\right)=\begin{bmatrix}\gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

the transformed $\overline{F}_{\mu\nu}=\Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu}F_{\alpha\beta}$. We should expect

$$\left(\overline{F}_{\mu\nu}\right)=\begin{bmatrix}0 & -\overline{E}_1 & -\overline{E}_2 & -\overline{E}_3\\ \overline{E}_1 & 0 & +\overline{B}_3 & -\overline{B}_2\\ \overline{E}_2 & -\overline{B}_3 & 0 & \overline{B}_1\\ \overline{E}_3 & \overline{B}_2 & -\overline{B}_1 & 0\end{bmatrix}.$$

By studying the equation, we can see that the electric and magnetic fields in the x-boosted frame relate to those in the original frame. We will find that the electric and magnetic fields in the boosted frame form a mixture of the $\vec{E}$ and $\vec{B}$.

Find $\overline{E}_i$ and $\overline{B}_j$ for $i,j=1,2,3$ in terms of E_i, B_j and γ,β.

$$\overline{F}_{\mu\nu}=\Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu}F_{\alpha\beta}=\Lambda^{\alpha}_{\ \mu}F_{\alpha\beta}\Lambda^{\beta}_{\ \nu}=\left(\Lambda F\Lambda\right)_{\cancel{\alpha\beta}\,\mu\nu}$$

$$\overline{F}_{\mu\nu}=\begin{bmatrix}\gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}0 & -E_1 & -E_2 & -E_3\\ E_1 & 0 & B_3 & -B_2\\ E_2 & -B_3 & 0 & B_1\\ E_3 & B_2 & -B_1 & 0\end{bmatrix}\cdot\begin{bmatrix}\gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\overline{F}_{\mu\nu}=\begin{bmatrix}-\gamma\beta E_1 & -\gamma E_1 & -\gamma E_2-\gamma\beta B_3 & -\gamma E_3+\gamma\beta B_2\\ \gamma E_1 & \gamma\beta E_1 & \gamma\beta E_2+\gamma B_3 & \gamma\beta E_3-\gamma B_2\\ E_2 & -B_3 & 0 & B_1\\ E_3 & B_2 & -B_1 & 0\end{bmatrix}\cdot\begin{bmatrix}\gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\overline{F}_{\mu\nu}=\begin{bmatrix}-\gamma^2\beta E_1+\gamma^2\beta E_1 & \gamma^2\beta^2E_1-\gamma^2E_1 & -\gamma E_2-\gamma\beta B_3 & -\gamma E_3+\gamma\beta B_2\\ \gamma^2E_1-\gamma^2\beta^2E_1 & -\gamma^2\beta E_1+\gamma^2\beta E_1 & \gamma\beta E_2+\gamma B_3 & \gamma\beta E_3-\gamma B_2\\ \gamma E_2+\gamma\beta B_3 & -\gamma\beta E_2-\gamma B_3 & 0 & B_1\\ \gamma E_3-\gamma\beta B_2 & -\gamma\beta E_3+\gamma B_2 & -B_1 & 0\end{bmatrix}$$

$$\overline{F}_{\mu\nu} = \begin{bmatrix} 0 & (\gamma^2\beta^2 - \gamma^2)E_1 & -\gamma(E_2 + \beta B_3) & -\gamma(E_3 - \beta B_2) \\ (\gamma^2 - \gamma^2\beta^2)E_1 & 0 & \gamma(B_3 + \beta E_2) & -\gamma(B_2 - \beta E_3) \\ \gamma(E_2 + \beta B_3) & -\gamma(B_3 + \beta E_2) & 0 & B_1 \\ \gamma(E_3 - \beta B_2) & \gamma(B_2 - \beta E_3) & -B_1 & 0 \end{bmatrix}$$

* Note: $\gamma = \dfrac{1}{\sqrt{1-\beta^2}} \Rightarrow \gamma^2 - \gamma^2\beta^2 = 1 \quad and \quad \gamma^2\beta^2 - \gamma^2 = -1$

$$\overline{F}_{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -\gamma(E_2 + \beta B_3) & -\gamma(E_3 - \beta B_2) \\ E_1 & 0 & \gamma(\beta E_2 + B_3) & -\gamma(B_2 - \beta E_3) \\ \gamma(E_2 + \beta B_3) & -\gamma(\beta E_2 + B_3) & 0 & B_1 \\ \gamma(E_3 - \beta B_2) & \gamma(B_2 - \beta E_3) & -B_1 & 0 \end{bmatrix}$$

Therefore,

$$\begin{aligned} \overline{E}_1 &= E_1 & \overline{B}_1 &= B_1 \\ \overline{E}_2 &= \gamma(E_2 + \beta B_3) & \overline{B}_2 &= \gamma(B_2 - \beta E_3) \\ \overline{E}_3 &= \gamma(E_3 - \beta B_2) & \overline{B}_3 &= \gamma(B_3 + \beta E_2) \end{aligned}$$

If we had correctly begun with

$$\overline{F}_{\mu\nu} = (\Lambda^{-1})^{\alpha}_{\mu} (\Lambda^{-1})^{\beta}_{\nu} F_{\alpha\beta}$$

we would have obtained,

$$\begin{aligned} \overline{E}_1 &= E_1 & \overline{B}_1 &= B_1 \\ \overline{E}_2 &= \gamma(E_2 - \beta B_3) & \overline{B}_2 &= \gamma(B_2 + \beta E_3) \\ \overline{E}_3 &= \gamma(E_3 + \beta B_2) & \overline{B}_3 &= \gamma(B_3 - \beta E_2) \end{aligned}$$

these are the correct rules, sorry bout the set-up.

MA 430 Homework, contd.

36. Given the matrix $A \in GL(3, \mathbb{R})$, show the determinant with regard to the wedge product. Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

We now have

$$(ae_1 + be_2 + ce_3) \wedge (de_1 + fe_2 + ge_3) \wedge (ge_1 + he_2 + ie_3)$$

Expanding Further:

$$\begin{aligned}
&= ae_1 \wedge (de_1 + ee_2 + fe_3) \wedge (ge_1 + he_2 + ie_3) + \\
&\quad be_2 \wedge (de_1 + ee_2 + fe_3) \wedge (ge_1 + he_2 + ie_3) + \\
&\quad ce_3 \wedge (de_1 + ee_2 + fe_3) \wedge (ge_1 + he_2 + ie_3) \\
&= ae_1 \wedge (ee_2 + fe_3) \wedge (he_2 + ie_3) + \\
&\quad be_2(de_1 + fe_3) \wedge (ge_1 + ie_3) + \\
&\quad ce_3 \wedge (de_1 + ee_2) \wedge (ge_1 + he_2) \\
&= aei\ e_1 \wedge e_2 \wedge e_3 + afh\ e_1 \wedge e_3 \wedge e_2 + \\
&\quad bdi\ e_2 \wedge e_1 \wedge e_3 + bfg\ e_2 \wedge e_3 \wedge e_1 + \\
&\quad cdh\ e_3 \wedge e_1 \wedge e_2 + ceg\ e_3 \wedge e_2 \wedge e_1 \\
&= \{a(ei - fh) - b(di - fg) + c(dh - eg)\} e_1 \wedge e_2 \wedge e_3
\end{aligned}$$

Therefore,

$$\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Remark: this is correct if we interpret e_1, e_2, e_3 as row vectors so that

$$e_1 A = (a, b, c) = ae_1 + be_2 + ce_3$$
$$e_2 A = (d, e, f) = de_1 + ee_2 + fe_3$$
$$e_3 A = (g, h, i) = ge_1 + he_2 + ie_3$$

and also,

$$e_1 A \wedge e_2 A \wedge e_3 A = \det(A)\, e_1 \wedge e_2 \wedge e_3$$

this follows Miller but not my notes.