37. ~~39.~~ Complete the discussion on pg. 113. That is show the following:

$$* dy = dz \wedge dx \qquad *(dz \wedge dx) = dy$$
$$* dz = dx \wedge dy \qquad *(dx \wedge dy) = dz$$

Need to show: $* dy = dz \wedge dx$

$$\begin{aligned}
* dy &= \frac{1}{p!}\frac{1}{(n-p)}\delta_j^2 \in_{jki} dx^k \wedge dx^i \\
&= \frac{1}{1!}\frac{1}{(3-1)} \in_{2ki} dx^k \wedge dx^i \\
&= \frac{1}{2}\left(\in_{231} dx^3 \wedge dx^1 + \in_{213} dx^1 \wedge dx^3\right) \\
&= \frac{1}{2}\left(\in_{123} dx^3 \wedge dx^1 + (-\in_{123})(-dx^3 \wedge dx^1)\right) \\
&= \frac{1}{2}\left(2\in_{123} dx^3 \wedge dx^1\right) \\
&= dx^3 \wedge dx^1 \\
&= dz \wedge dx
\end{aligned}$$

Need to show: $* dz = dx \wedge dy$

$$\begin{aligned}
* dz &= \frac{1}{p!}\frac{1}{(n-p)}\delta_k^3 \in_{kij} dx^i \wedge dx^j \\
&= \frac{1}{1!}\frac{1}{(3-1)} \in_{3ij} dx^i \wedge dx^j \\
&= \frac{1}{2}\left(\in_{312} dx^1 \wedge dx^2 + \in_{321} dx^2 \wedge dx^1\right) \\
&= \frac{1}{2}\left(\in_{123} dx^1 \wedge dx^2 + (-\in_{123})(-dx^1 \wedge dx^2)\right) \\
&= \frac{1}{2}\left(2\in_{123} dx^1 \wedge dx^2\right) \\
&= dx^1 \wedge dx^2 \\
&= dx \wedge dy
\end{aligned}$$

MA 430 Homework, contd.

38. Determine if $\dim(\Lambda^p(V)) = \frac{n!}{(n-p)!p!}$. We know that $\dim(V) = n$.

When $V = \mathbb{R}^3, \dim(V) = 3$. We have

$$\dim(\Lambda^1(V)) = \frac{3!}{(3-1)!1!} = \frac{3!}{2!} = 3$$

$$\dim(\Lambda^2(V)) = \frac{3!}{(3-2)!2!} = \frac{3!}{2!} = 3$$

$$\therefore \dim(\Lambda^1(V)) = \dim(\Lambda^2(V)) = \dim(V) \text{ for } n = 3$$

Consider $n \neq 3$. When $V = \mathbb{R}$, $\dim(V) = 1$.

$$\dim(\Lambda^1(V)) = \frac{1!}{(1-1)!1!} = 1$$

$$\dim(\Lambda^2(V)) = \frac{1!}{(1-2)!1!} = \frac{1}{-1 \cdot 2!} = -\frac{1}{2}$$

$$\therefore \dim(\Lambda^1(V)) \neq \dim(\Lambda^2(V))$$

I suppose we should have remarked the formula only applies for $p = 0, 1, 2, \ldots, n$ here $p = 2 > 1 = n$.

When $V = \mathbb{R}^2$, $\dim(V) = 2$

$$\dim(\Lambda^1(V)) = \frac{2!}{(2-1)!1!} = \frac{2!}{1!} = 2$$

$$\dim(\Lambda^2(V)) = \frac{2!}{(2-2)!2!} = \frac{2!}{2!} = 1$$

$$\therefore \dim(\Lambda^1(V)) \neq \dim(\Lambda^2(V))$$

When $V = \mathbb{R}^4$, $\dim(V) = 4$

$$\dim(\Lambda^1(V)) = \frac{4!}{(4-1)!1!} = \frac{4!}{3!} = 4$$

$$\dim(\Lambda^1(V)) = \frac{4!}{(4-2)!2!} = \frac{4!}{2! \cdot 2!} = 6$$

$$\therefore \dim(\Lambda^1(V)) \neq \dim(\Lambda^2(V))$$

Only when $n = 3$ does $\dim(\Lambda^1(V)) = \dim(\Lambda^2(V)) = \dim V$, otherwise $\dim(\Lambda^1(V)) \neq \dim(\Lambda^2(V))$.

39. (a) Find BOB when $V = \mathbb{R}^3$ and $g_{ij} = \delta_{ij}$

$$*(dx) = dy \wedge dz \quad \& \quad *(dy \wedge dz) = dx$$

$$\therefore **dx = dx \rightarrow \text{BOB} = 0$$

$$*1 = dx \wedge dy \wedge dz \quad \& \quad *(dx \wedge dy \wedge dz) = 1$$

$$\therefore **1 = 1 \rightarrow \text{BOB} = 0$$

$$**\alpha = (-1)^{p(n-1)+s}\alpha = (-1)^{3+1}\alpha = \alpha$$ (it follows my claim)

(b) Find BOB when $V = \mathbb{R}^4$ and $g_{\mu\nu} = \eta_{\mu\nu}$

$$*(dt) = -dx\wedge dy\wedge dz \quad \& \quad *(-dx\wedge dy\wedge dz) = -(-dt) = dt$$

$$\therefore **dt = dt \to \text{BOB} = 0$$

Same for $**dx^\mu = dx^\mu$ for $\mu = 0,1,2,3,\cdots$

$$*(dz\wedge dt) = dx\wedge dy \quad \& \quad *(-dx\wedge dy) = -(-dz\wedge dt) = dz\wedge dt$$

$$\therefore **(dz\wedge dt) = dz\wedge dt \to \text{BOB} = 0$$

Same for $**(dx^\mu\wedge dx^\nu) = dx^\mu\wedge dx^\nu$, for $\mu,\nu = 0,1,2,3$ and $\mu\neq\nu$.

$$*1 = dt\wedge dx\wedge dy\wedge dz \quad \& \quad *dt\wedge dx\wedge dy\wedge dz = 1$$

$$\therefore **1 = 1 \to \text{BOB} = 0$$

not quite.

(c) This case is identical to case (b). $\Rightarrow$ Since the metrics are the same (?)

-2

41. Need to show that $\dim(\Lambda(V)) = 2^{\dim(V)}$.

We know that $\dim(V) = n$, so $2^{\dim(V)} = 2^n$. We also know $\dim(\Lambda^p(V)) = \frac{n!}{(n-p)!p!}$. To find $\dim(\Lambda(V))$, we need to sum $\dim(\Lambda^0(V)) + \dim^1(\Lambda(V)) + \cdots + \dim(\Lambda^n(V))$.

(a) $n = 0: \dim(\Lambda(V)) = \dim(\Lambda^0(V)) = \frac{0!}{(0-0)!0!} = 1 = 2^0$

(b) $n = 1:$

$$\begin{aligned}\dim(\Lambda(V)) &= \dim(\Lambda^0(V)) + \dim(\Lambda^1(V))\\ &= \frac{1!}{(1-0)!0!} + \frac{1!}{(1-1)!1!}\\ &= 1+1 = 2-2^1\end{aligned}$$

(c) $n = 2:$

$$\begin{aligned}\dim(\Lambda(V)) &= \dim(\Lambda^0(V)) + \dim(\Lambda^1(V)) + \dim(\Lambda^2(V))\\ &= \frac{2!}{(2-0)!0!} + \frac{2!}{(2-1)!1!} + \frac{2!}{(2-2)!2!}\\ &= 1+2+1\\ &= 4 = 2^2\end{aligned}$$

(d) $n = 3:$

$$\begin{aligned}\dim(\Lambda(V)) &= \dim(\Lambda^0(V)) + \dim(\Lambda^1(V)) + \dim(\Lambda^2(V)) + \dim(\Lambda^3(V))\\ &= \frac{3!}{(3-0)!0!} + \frac{3!}{(3-1)!1!} + \frac{3!}{(3-2)!2!} + \frac{3!}{(3-3)!3!}\\ &= 1+3+3+1\\ &= 8 = 2^3\end{aligned}$$

And so on and so on and so on and so forth and the rest is silence.

$$\therefore \dim(\Lambda(V)) = 2^{\dim(V)}$$

40. Verify the claim that for vectors $A, B \in R^3$: $\omega_A \wedge \omega_B = \Phi_{A\times B}$

Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$.

First we will evaluate $\omega_A \wedge \omega_B$.

Given A and B, we have $\omega_A = a_1dx + a_2dy + a_3dz$ and $\omega_B = b_1dx + b_2dy + b_3dz$

$$\omega_A \wedge \omega_B = (a_1dx + a_2dy + a_3dz) \wedge (b_1dx + b_2dy + b_3dz)$$

$$= a_1dx \wedge (b_1dx + b_2dy + b_3dz) + a_2dy \wedge (b_1dx + b_2dy + b_3dz) + a_3dz \wedge (b_1dx + b_2dy + b_3dz)$$

$$\begin{aligned} &= a_1dx \wedge b_1dx + a_1dx \wedge b_2dy + a_1dx \wedge b_3dz \\ &+ a_2dy \wedge b_1dx + a_2dy \wedge b_2dy + a_2dy \wedge b_3dz \\ &+ a_3dz \wedge b_1dx + a_3dz \wedge b_2dy + a_3dz \wedge b_3dz \end{aligned}$$

$$= a_1b_2dx \wedge dy + a_1b_3dx \wedge dz + a_2b_1dy \wedge dx + a_2b_3dy \wedge dz + a_3b_1dz \wedge dx + a_3b_2dz \wedge dy$$

$$= a_1b_2dx \wedge dy - a_1b_3dz \wedge dx - a_2b_1dx \wedge dy + a_2b_3dy \wedge dz + a_3b_1dz \wedge dx - a_3b_2dy \wedge dz$$

$$= (a_1b_2 - a_2b_1)dx \wedge dy + (a_3b_1 - a_1b_3)dz \wedge dx + (a_2b_3 - a_3b_2)dy \wedge dz$$

Next, we will evaluate $\Phi_{A\times B}$.

To do so, we need to find $\vec{A}\times\vec{B}$:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

As a result, $\Phi_{A\times B} = (a_1b_2 - a_2b_1)dx \wedge dy + (a_3b_1 - a_1b_3)dz \wedge dx + (a_2b_3 - a_3b_2)dy \wedge dz$

Therefore, we see that $\omega_A \wedge \omega_B = \Phi_{A\times B}$.

#37

$$\dim(\Lambda^p(V)) = \frac{n!}{(n-p)!\,p!} \quad p=0..n$$
p forms $\xrightarrow{*}$ $(n-p)$ forms

0 If $V \in \mathbb{R}^0$, $n=0$.

$p=0$ $\dim(\Lambda^0(V)) = \frac{0!}{(0-0)!\,0!} = 1$ 0-forms $\xrightarrow{*}$ 0-forms

---

1 If $V \in \mathbb{R}^1$, $n=1$

$p=0$ $\dim(\Lambda^0(V)) = \frac{1!}{(1-0)!\,0!} = 1$ 0-forms $\xrightarrow{*}$ 1-forms

$p=1$ $\dim(\Lambda^1(V)) = \frac{1!}{(1-1)!\,1!} = 1$ 1-forms $\xrightarrow{*}$ 0-forms

---

2 If $V \in \mathbb{R}^2$, $n=2$

$p=0$ $\dim(\Lambda^0(V)) = \frac{2!}{(2-0)!\,0!} = 1$ 0-forms $\xrightarrow{*}$ 2-forms

$p=1$ $\dim(\Lambda^1(V)) = \frac{2!}{(2-1)!\,1!} = 2$ 1-forms $\xrightarrow{*}$ 1-forms

$p=2$ $\dim(\Lambda^2(V)) = \frac{2!}{(2-2)!\,2!} = 1$ 2-forms $\xrightarrow{*}$ 0-forms

---

3 If $V \in \mathbb{R}^3$, $n=3$

$p=0$ $\dim(\Lambda^0(V)) = \frac{3!}{(3-0)!\,0!} = 1$ 0-forms $\xrightarrow{*}$ 3-forms

$p=1$ $\dim(\Lambda^1(V)) = \frac{3!}{(3-1)!\,1!} = 3$ 1-forms $\xrightarrow{*}$ 2-forms

$p=2$ $\dim(\Lambda^2(V)) = \frac{3!}{(3-2)!\,2!} = 3$ 2-forms $\xrightarrow{*}$ 1-forms

$p=3$ $\dim(\Lambda^3(V)) = \frac{3!}{(3-3)!\,3!} = 1$ 3-forms $\xrightarrow{*}$ 0-forms

4 If $V \in \mathbb{R}^4$, $n=4$

$p=0$ $\dim(\Lambda^0(V)) = \frac{4!}{(4-0)!\,0!} = 1$ 0-forms $\xrightarrow{*}$ 4-forms

$p=1$ $\dim(\Lambda^1(V)) = \frac{4!}{(4-1)!\,1!} = 4$ 1-forms $\xrightarrow{*}$ 3-forms

$p=2$ $\dim(\Lambda^2(V)) = \frac{4!}{(4-2)!\,2!} = 6$ 2-forms $\xrightarrow{*}$ 2-forms

$p=3$ $\dim(\Lambda^3(V)) = \frac{4!}{(4-3)!\,3!} = 4$ 3-forms $\xrightarrow{*}$ 1-forms

$p=4$ $\dim(\Lambda^4(V)) = \frac{4!}{(4-4)!\,4!} = 1$ 4-forms $\xrightarrow{*}$ 0-forms

---

5 If $V \in \mathbb{R}^5$, $n=5$

$p=0$ $\dim(\Lambda^0(V)) = \frac{5!}{(5-0)!\,0!} = 1$ 0-forms $\xrightarrow{*}$ 5-forms

$p=1$ $\dim(\Lambda^1(V)) = \frac{5!}{(5-1)!\,1!} = 5$ 1-forms $\xrightarrow{*}$ 4-forms

$p=2$ $\dim(\Lambda^2(V)) = \frac{5!}{(5-2)!\,2!} = 10$ 2-forms $\xrightarrow{*}$ 3-forms

$p=3$ $\dim(\Lambda^3(V)) = \frac{5!}{(5-3)!\,3!} = 10$ 3-forms $\xrightarrow{*}$ 2-forms

$p=4$ $\dim(\Lambda^4(V)) = \frac{5!}{(5-4)!\,4!} = 5$ 4-forms $\xrightarrow{*}$ 1-forms

$p=5$ $\dim(\Lambda^5(V)) = \frac{5!}{(5-5)!\,5!} = 1$ 5-forms $\xrightarrow{*}$ 0-forms

From these examples we noticed that if $V \in \mathbb{R}^3$ ($n=3$), then $\dim(\Lambda^1(V)) = \dim(\Lambda^2(V))$. This is the only case where two consecutive p's (where $0 \le p \le n$), that the $\dim(\Lambda^p(V)) = \dim(\Lambda^{p+1}(V))$, which is also equal to n, the dimension of V.

We also noticed that as the dimension of V, (n), increases by 1 we get the next row of Pascal's triangle, as shown below.

KEY
— Pascal's Triangle
p-forms → (n-p)-forms

$n=0$	$\overset{0\to 0}{1}$					
$n=1$	$\overset{0\to 1}{1}$	$\overset{1\to 0}{1}$				
$n=2$	$\overset{0\to 2}{1}$	$\overset{1\to 1}{2}$	$\overset{2\to 0}{1}$			
$n=3$	$\overset{0\to 3}{1}$	$\overset{1\to 2}{3}$	$\overset{2\to 1}{3}$	$\overset{3\to 0}{1}$		
$n=4$	$\overset{0\to 4}{1}$	$\overset{1\to 3}{4}$	$\overset{2\to 2}{6}$	$\overset{3\to 1}{4}$	$\overset{4\to 0}{1}$	
$n=5$	$\overset{0\to 5}{1}$	$\overset{1\to 4}{5}$	$\overset{2\to 3}{10}$	$\overset{3\to 2}{10}$	$\overset{4\to 1}{5}$	$\overset{5\to 0}{1}$

$\vdots$

From this we realized that $\dim(\Lambda^0(V)) = \dim(\Lambda^n(V)) = 1$.

Also that $\dim(\Lambda^1(V)) = \dim(\Lambda^{n-1}(V)) = n$.

Only in the 3-dim (we think!) there is an isomorphism so that:

$V \in \mathbb{R}^3,\ n=3$

$\dim(\Lambda^1(V)) = \frac{3!}{(3-1)!\,1!} = 3$ | $\dim(\Lambda^{3-1}(V)) = \frac{3!}{(3-2)!\,2!} = 3$

so $\Lambda^1(V) \in V \quad (n=3)$ | so $\Lambda^2(V) \in V \quad (n=3)$

#38

Hodge duality

$$\Lambda^p(V) \xrightarrow{*} \Lambda^{n-p}(V) \xrightarrow{*} \Lambda^p(V)$$

$$**\alpha = (-1)^{BOB}\alpha$$

i) Suppose $V \in \mathbb{R}^3$ $(n=3)$ and the metric is $g_{ij} = \delta_{ij}$

(p=0) Then the $\Lambda^0(V) \xrightarrow{*} \Lambda^{3-0}(V) \xrightarrow{*} \Lambda^0(V)$ is:

$$1 \xrightarrow{*} dx \wedge dy \wedge dz \xrightarrow{*} 1 \qquad **\alpha = \alpha$$

"BOB" is 0 or <u>even</u>

(p=1) The $\Lambda^1(V) \xrightarrow{*} \Lambda^{3-1}(V) \xrightarrow{*} \Lambda^1(V)$ are:

$$\left.\begin{array}{l} dx \xrightarrow{*} dy \wedge dz \xrightarrow{*} dx \\ dy \xrightarrow{*} dz \wedge dx \xrightarrow{*} dy \\ dz \xrightarrow{*} dx \wedge dy \xrightarrow{*} dz \end{array}\right\} **\alpha = \alpha$$

"BOB" is 0 or <u>even</u>

(p=2) The $\Lambda^2(V) \xrightarrow{*} \Lambda^{3-2}(V) \xrightarrow{*} \Lambda^2(V)$ are:

$$\left.\begin{array}{l} dx \wedge dy \xrightarrow{*} dz \xrightarrow{*} dx \wedge dy \\ dy \wedge dz \xrightarrow{*} dx \xrightarrow{*} dy \wedge dz \\ dz \wedge dx \xrightarrow{*} dy \xrightarrow{*} dz \wedge dx \end{array}\right\} **\alpha = \alpha$$

"BOB" is 0 or <u>even</u>

(p=3) The $\Lambda^3(V) \xrightarrow{*} \Lambda^{3-3}(V) \xrightarrow{*} \Lambda^3(V)$ is:

$$dx \wedge dy \wedge dz \xrightarrow{*} 1 \xrightarrow{*} dx \wedge dy \wedge dz \qquad **\alpha = \alpha$$

"BOB" is 0 or <u>even</u>

$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \; 1 \; 1 \end{bmatrix}$

ii) Suppose $V \in \mathbb{R}^4$ $(n=4)$ and the metric is $g_{\mu\nu} = \eta_{\mu\nu}$

$(p=0)$ $\Lambda^0(V) \xrightarrow{*} \Lambda^{4-0}(V) \xrightarrow{*} \Lambda^0(V)$ form:

$$1 \xrightarrow{*} -dt \wedge dx \wedge dy \wedge dz \xrightarrow{*} -1 \qquad **\alpha = -\alpha$$

"BOB" is <u>odd</u>

$(p=1)$ $\Lambda^1(V) \xrightarrow{*} \Lambda^{4-1}(V) \xrightarrow{*} \Lambda^1(V)$ forms:

$$\left.\begin{array}{l} dt \xrightarrow{*} -dx \wedge dy \wedge dz \xrightarrow{*} dt \\ dx \xrightarrow{*} -dy \wedge dz \wedge dt \xrightarrow{*} dx \\ dy \xrightarrow{*} -dz \wedge dx \wedge dt \xrightarrow{*} dy \\ dz \xrightarrow{*} -dx \wedge dy \wedge dt \xrightarrow{*} dz \end{array}\right\} **\alpha = \alpha$$

"BOB" is 0 or <u>even</u>

$(p=2)$ $\Lambda^2(V) \xrightarrow{*} \Lambda^{4-2}(V) \xrightarrow{*} \Lambda^2(V)$ forms:

$$\left.\begin{array}{l} dx \wedge dt \xrightarrow{*} dy \wedge dz \xrightarrow{*} -dx \wedge dt \\ dy \wedge dt \xrightarrow{*} dz \wedge dx \xrightarrow{*} -dy \wedge dt \\ dz \wedge dt \xrightarrow{*} dx \wedge dy \xrightarrow{*} -dz \wedge dt \\ dx \wedge dy \xrightarrow{*} -dz \wedge dt \xrightarrow{*} -dx \wedge dy \\ dy \wedge dz \xrightarrow{*} -dx \wedge dt \xrightarrow{*} -dy \wedge dz \\ dz \wedge dx \xrightarrow{*} -dy \wedge dt \xrightarrow{*} -dz \wedge dx \end{array}\right\} **\alpha = -\alpha$$

"BOB" is <u>odd</u>

$(p=3)$ $\Lambda^3(V) \xrightarrow{*} \Lambda^{4-3}(V) \xrightarrow{*} \Lambda^3(V)$ forms:

I rewrote these diff. from notes p117

$$\left.\begin{array}{l} dx \wedge dy \wedge dz \xrightarrow{*} -dt \xrightarrow{*} dx \wedge dy \wedge dz \\ dt \wedge dy \wedge dz \xrightarrow{*} -dx \xrightarrow{*} dt \wedge dy \wedge dz \\ dt \wedge dz \wedge dx \xrightarrow{*} -dy \xrightarrow{*} dt \wedge dz \wedge dx \\ dt \wedge dx \wedge dy \xrightarrow{*} -dz \xrightarrow{*} dt \wedge dx \wedge dy \end{array}\right\} **\alpha = \alpha$$

"BOB" is 0 or <u>even</u>

$(p=4)$ $\Lambda^4(V) \xrightarrow{*} \Lambda^{4-4}(V) \xrightarrow{*} \Lambda^4(V)$ form:

$$dt \wedge dx \wedge dy \wedge dz \xrightarrow{*} 1 \xrightarrow{*} -dt \wedge dx \wedge dy \wedge dz \qquad **\alpha = -\alpha$$

"BOB" is <u>odd</u>

$$-\eta_{\mu\nu} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$$

iii) Suppose $V \in \mathbb{R}^4$ $(n=4)$ and the metric is $g_{\mu\nu} = \eta_{\mu\nu}$

(p=0) $\Lambda^0(V) \xrightarrow{*} \Lambda^{4-0}(V) \xrightarrow{*} \Lambda^0(V)$ form:

$1 \xrightarrow{*} -dt \wedge dx \wedge dy \wedge dz \xrightarrow{*} -1$ $\quad **\alpha = -\alpha$

"BOB" is <u>odd</u>

(p=1) $\Lambda^1(V) \xrightarrow{*} \Lambda^{4-1}(V) \xrightarrow{*} \Lambda^1(V)$ forms:

$$\left.\begin{array}{l} dt \xrightarrow{*} dx \wedge dy \wedge dz \xrightarrow{*} -dt \\ dx \xrightarrow{*} dy \wedge dz \wedge dt \xrightarrow{*} -dx \\ \checkmark\ dy \xrightarrow{*} dz \wedge dy \wedge dt \xrightarrow{*} -dy \\ dz \xrightarrow{*} dx \wedge dy \wedge dt \xrightarrow{*} -dz \end{array}\right\} \begin{array}{l} **\alpha = -\alpha \\ \text{"BOB" is \underline{even}} \end{array}$$

(p=2) $\Lambda^2(V) \xrightarrow{*} \Lambda^{4-2}(V) \xrightarrow{*} \Lambda^2(V)$ forms:

$$\left.\begin{array}{l} dx \wedge dt \xrightarrow{*} dy \wedge dz \xrightarrow{*} -dx \wedge dt \\ dy \wedge dt \xrightarrow{*} dz \wedge dx \xrightarrow{*} -dy \wedge dt \\ dz \wedge dt \xrightarrow{*} dx \wedge dy \xrightarrow{*} -dz \wedge dt \\ dx \wedge dy \xrightarrow{*} -dz \wedge dt \xrightarrow{*} -dx \wedge dy \\ dy \wedge dz \xrightarrow{*} -dx \wedge dt \xrightarrow{*} -dy \wedge dz \\ dz \wedge dx \xrightarrow{*} -dy \wedge dt \xrightarrow{*} -dz \wedge dx \end{array}\right\} \begin{array}{l} **\alpha = -\alpha \\ \text{"BOB" is \underline{odd}} \end{array}$$

(p=3) $\Lambda^3(V) \xrightarrow{*} \Lambda^{4-3}(V) \xrightarrow{*} \Lambda^3(V)$ forms:

$$\left.\begin{array}{l} dx \wedge dy \wedge dz \xrightarrow{*} dt \xrightarrow{*} -dx \wedge dy \wedge dz \\ dt \wedge dy \wedge dz \xrightarrow{*} dx \xrightarrow{*} -dt \wedge dy \wedge dz \\ \checkmark\ dt \wedge dz \wedge dx \xrightarrow{*} dy \xrightarrow{*} -dt \wedge dz \wedge dx \\ dt \wedge dx \wedge dy \xrightarrow{*} dz \xrightarrow{*} -dt \wedge dx \wedge dy \end{array}\right\} \begin{array}{l} **\alpha = \alpha \\ \text{"BOB" is \underline{even}} \end{array}$$

(p=4) $\Lambda^4(V) \xrightarrow{*} \Lambda^{4-4}(V) \xrightarrow{*} \Lambda^4(V)$ form:

$dt \wedge dx \wedge dy \wedge dz \xrightarrow{*} -1 \xrightarrow{*} -dt \wedge dx \wedge dy \wedge dz$

$**\alpha = -\alpha$

"BOB" is <u>odd</u>

As suggested in class "BOB" $= p(n-1) + S$, where S is the signiture of the metric. The signiture is defined as the number of "-1"'s in the metric matrix.

To check that this is consistant with parts i, ii, & iii:

i) $V \in \mathbb{R}^3$ $(n=3)$; $g_{ij} = \delta_{ij}$ $(s=0)$ $g_{ij} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$

$(p=0)$ "BOB" $= 0(3-1)+0 = 0$ ✓
$(p=1)$ "BOB" $= 1(3-1)+0 = 2$ even ✓
$(p=2)$ "BOB" $= 2(3-1)+0 = 4$ even ✓
$(p=3)$ "BOB" $= 3(3-1)+0 = 6$ even ✓

ii) $V \in \mathbb{R}^4$ $(n=4)$; $g_{\mu\nu} = \eta_{\mu\nu}$ $(s=1)$; $\eta_{\mu\nu} = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$

$(p=0)$ "BOB" $= 0(4-1)+1 = 1$ odd ✓
$(p=1)$ "BOB" $= 1(4-1)+1 = 4$ even ✓
$(p=2)$ "BOB" $= 2(4-1)+1 = 7$ odd ✓
$(p=3)$ "BOB" $= 3(4-1)+1 = 10$ even ✓
$(p=4)$ "BOB" $= 4(4-1)+1 = 13$ odd ✓

iii) $V \in \mathbb{R}^4$ $(n=4)$; $g_{\mu\nu} = -\eta_{\mu\nu}$ $(s=3)$; $-\eta_{\mu\nu} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$

$(p=0)$ "BOB" $= 0(4-1)+3 = 3$ odd ✓
$(p=1)$ "BOB" $= 1(4-1)+3 = 6$ even ✓
$(p=2)$ "BOB" $= 2(4-1)+3 = 9$ odd ✓
$(p=3)$ "BOB" $= 3(4-1)+3 = 12$ even ✓
$(p=4)$ "BOB" $= 4(4-1)+3 = 15$ odd ✓

#39

$$*(dy) = \left(\frac{1}{p!}\right)\left(\frac{1}{(n-p)!}\right) \delta^2_j \epsilon_{ijk}\, dx^i \wedge dx^k$$
$$= \left(\frac{1}{1!}\right)\left(\frac{1}{2!}\right) \epsilon_{i2k}\, dx^i \wedge dx^k$$
$$= \frac{1}{2}\left(\epsilon_{123}\, dx^1 \wedge dx^3 + \epsilon_{321}\, dx^3 \wedge dx^1\right)$$
$$= \frac{1}{2}\left(dx^1 \wedge dx^3 - (-dx^1 \wedge dx^3)\right)$$
$$= dx^1 \wedge dx^3$$
$$= dx \wedge dz$$

$$*(dz) = \left(\frac{1}{1!}\right)\left(\frac{1}{2!}\right) \delta^3_k \epsilon_{ijk}\, dx^i \wedge dx^j$$
$$= \frac{1}{2}\left(\epsilon_{123}\, dx^1 \wedge dx^2 + \epsilon_{213}\, dx^2 \wedge dx^1\right)$$
$$= \frac{1}{2}\left(dx^1 \wedge dx^2 - (-dx^1 \wedge dx^2)\right)$$
$$= dx^1 \wedge dx^2$$
$$= dx \wedge dy$$

$$*(dz \wedge dx) = \left(\frac{1}{2!}\right)\left(\frac{1}{1!}\right) 2\, \delta^3_k \delta^1_i \epsilon_{ijk}\, dx^j$$
$$= \left(\frac{1}{2}\right) 2\, \epsilon_{1j3}\, dx^j$$
$$= \epsilon_{123}\, dx^2$$
$$= dx^2$$
$$= dy$$

$$*(dx \wedge dy) = \left(\frac{1}{2}\right) 2\, \delta^1_i \delta^2_j \epsilon_{ijk}\, dx^k$$
$$= \epsilon_{12k}\, dx^k$$
$$= \epsilon_{123}\, dx^3$$
$$= dx^3$$
$$= dz$$