The Final MA 430 Homework

49. Using the generalized Stoke's Theorem, show that

$$\text{vol}(M) = \int_{\partial M} x^1 dx^2 \wedge dx^3 \wedge \cdots \wedge dx^n$$

Let $\beta = x^1 dx^2 \wedge dx^3 \wedge \cdots \wedge dx^n$. Through the generalized Stoke's Theorem, we know that

$$\int_{\partial M} \beta = \int_M d\beta$$

If $d\beta$ is a volume form, then we're good to go, where the volume form is $dx^1 \wedge dx^2 \wedge \cdots dx^n$. Well, $d\beta = dx^1 \wedge (dx^2 \wedge \cdots \wedge dx^n)$ is a volume form, so

$$\int_M (\text{volume form}) = \text{vol}(M) \Rightarrow \text{vol}(M) = \int_{\partial M} x^1 dx^2 \wedge dx^3 \wedge \cdots \wedge dx^n$$

50. Explain why

$$\int_M f dx \wedge dy \wedge dz = \iiint_{D_3} f \; dx \; dy \; dz$$

where $X : D_3 \to M$ is a parametrization of M by the cartesian coordinates x, y, and z.

Start by letting $\gamma = f dx \wedge dy \wedge dz$, which in turn

$$\gamma = \frac{1}{6} f \epsilon_{ijk} dx^i \wedge dx^j \wedge dx^k$$

So

$$\begin{aligned}\int_M \gamma &= \iiint \gamma_{ijk} \frac{\partial x^i}{\partial x} \frac{\partial x^j}{\partial y} \frac{\partial x^k}{\partial z} dx \; dy \; dz \\ &= \iiint \gamma_{ijk} \delta^i_1 \delta^j_2 \delta^k_3 dx \; dy \; dz \\ &= \iiint \gamma_{ijk} dx \; dy \; dz \\ &= \iiint f dx \; dy \; dz\end{aligned}$$

51. Consider $D^2 \subset R^2$ a dish of radius R. That is $D^2 = \{(x,y) \mid x^2 + y^2 \le R^2\}$. Note $\partial D^2 = S^1 = \{(x,y) \mid x^2 + y^2 = R^2\}$. The natural parameterization of D^2 is $x = r\cos\phi$, $y = r\sin\phi$.

(a) Show $dx \wedge dy = r\,dr \wedge d\phi$

$$dx = \frac{dx}{dr}dr + \frac{dx}{d\phi}d\phi = \cos\phi\,dr - r\sin\phi\,d\phi$$
$$dy = \frac{dy}{dr}dr + \frac{dy}{d\phi}d\phi = \sin\phi\,dr + r\cos\phi\,d\phi$$

$$\begin{aligned}
dx \wedge dy &= (\cos\phi\,dr - r\sin\phi\,d\phi)\wedge(\sin\phi\,dr + r\cos\phi\,d\phi) \\
&= r\cos^2\phi\,dr \wedge d\phi - r\sin^2\phi\,d\phi \wedge dr \\
&= r\cos^2\phi\,dr \wedge d\phi + r\sin^2\phi\,dr \wedge d\phi \\
&= (r\cos^2\phi + r\sin^2\phi)dr \wedge d\phi \\
&= (r(\cos^2\phi + \sin^2\phi))dr \wedge d\phi \\
&= r\,dr \wedge d\phi
\end{aligned}$$

(b) Show $vol(D^2) = \pi R^2$ by calculating $\int_{D^2} vol_2$.

$$Vol(D^2) = \int_{D^2} vol_2 = \int_{D^2} dx \wedge dy = \int_{D^2} r dr \wedge d\phi = \int_0^R r dr \int_0^{2\pi} d\phi = \left.\frac{r^2}{2}\right|_0^R \left.\phi\right|_0^{2\pi} = \frac{R^2}{2}\cdot 2\pi = \pi R^2$$

(c) Alternatively consider $\beta = xdy$. Use the fact that $\int_{D^2} \partial\beta = \int_{\partial D^2} \beta$ to show $vol(D^2) = \pi R^2$.

$$Vol(D^2) = \int_{D^2} \partial\beta = \int_{\partial D^2} \beta = \int_{\partial D^2} xdy = \int_{\partial D^2} (r\cos\phi)(\sin\phi\,dr + r\cos\phi\,d\phi) \qquad \{\text{Note: } dr = 0\}$$

$$= \int_{\partial D^2} (r\cos\phi)(r\cos\phi\,d\phi) = \int_{\partial D^2} r^2\cos^2\phi\,d\phi = r^2 \int_{\partial D^2} \cos^2\phi\,d\phi$$

$$= r^2 \int_0^{2\pi} \frac{1}{2}(1 + \cos 2\phi)d\phi = r^2 \int_0^{2\pi} \frac{1}{2} d\phi = r^2 \left.\frac{\phi}{2}\right|_0^{2\pi} = \pi R^2 \qquad \{\text{Note: } \int_0^{2\pi} \cos 2\phi = 0\}$$

(d) Argue by symmetry that $area(D^2) = 2\pi R$.

$$vol(D^2) = \pi r^2$$
$$d(vol(D^2)) = d(\pi r^2)$$
$$d(vol(D^2)) = 2\pi r\, dr$$
$$\frac{d(vol(D^2))}{dr} = 2\pi r$$
$$area(D^2) = 2\pi R$$

52. Consider $D^3 = \{(x,y,z) \mid x^2 + y^2 + z^2 \le R^2\} \subset R^3$. Notice that $\partial D^3 = S^2 = \{(x,y,z) \mid x^2 + y^2 + z^2 = R^2\}$. The natural parameterization of D^3 is $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\theta$ where $0 \le r \le R$, $0 \le \phi \le 2\pi$, $0 \le \theta \le \pi$.

(a) Show $dx \wedge dy \wedge dz = r^2 \sin\theta\, dr \wedge d\theta \wedge d\phi$

$$dx = \frac{dx}{dr}dr + \frac{dx}{d\phi}d\phi + \frac{dx}{d\theta}d\theta = \cos\phi\sin\theta\, dr - r\sin\phi\sin\theta\, d\phi + r\sin\phi\cos\theta\, d\theta$$
$$dy = \frac{dy}{dr}dr + \frac{dy}{d\phi}d\phi + \frac{dy}{d\theta}d\theta = \sin\phi\sin\theta\, dr + r\cos\phi\sin\theta\, d\phi + r\sin\phi\cos\theta\, d\theta$$
$$dz = \frac{dz}{dr}dr + \frac{dz}{d\phi}d\phi + \frac{dz}{d\theta}d\theta = \cos\theta\, dr - r\sin\theta\, d\theta$$

We'll start by finding $dx \wedge dy$.

$$dx \wedge dy = (\cos\phi\sin\theta\, dr - r\sin\phi\sin\theta\, d\phi + r\sin\phi\cos\theta\, d\theta) \wedge (\sin\phi\sin\theta\, dr + r\cos\phi\sin\theta\, d\phi + r\sin\phi\cos\theta\, d\theta)$$

$$= r\cos^2\phi\sin^2\theta\, dr \wedge d\phi + r\cos\phi\sin\theta\sin\phi\cos\theta\, dr \wedge d\theta - r\sin^2\phi\sin^2\theta\, d\phi \wedge dr - r^2\sin^2\phi\sin\theta\cos\theta\, d\phi \wedge d\theta + r\cos\phi\sin\theta\sin\phi\cos\theta\, d\theta \wedge dr + r^2\cos^2\phi\sin\theta\cos\theta\, d\theta \wedge d\phi$$

$$= (r\cos^2\phi\sin^2\theta + r\sin^2\phi\sin^2\theta)\, dr \wedge d\phi + (r\cos\phi\sin\theta\sin\phi\cos\theta - r\cos\phi\sin\theta\sin\phi\cos\theta)\, dr \wedge d\theta + (r^2\sin^2\phi\sin\theta\cos\theta + r^2\cos^2\phi\sin\theta\cos\theta)\, d\theta \wedge d\phi$$

$$= (r\sin^2\theta(\cos^2\phi + \sin^2\phi))\, dr \wedge d\phi + (r^2\sin\theta\cos\theta(\sin^2\phi + \cos^2\phi))\, d\theta \wedge d\phi$$

$$= r\sin^2\theta\, dr \wedge d\phi + r^2\sin\theta\cos\theta\, d\theta \wedge d\phi$$

Now we'll find $dx \wedge dy \wedge dz$.

$$\begin{aligned} dx \wedge dy \wedge dz &= \left(r\sin^2\theta\, dr \wedge d\phi + r^2 \sin\theta\cos\theta\, d\theta \wedge d\phi\right) \wedge \left(\cos\theta\, dr - r\sin\theta\, d\theta\right) \\ &= -r^2 \sin^3\theta\, dr \wedge d\phi \wedge d\theta + r^2 \sin\theta \cos^2\theta\, d\theta \wedge d\phi \wedge dr \\ &= r^2 \sin^3\theta\, dr \wedge d\theta \wedge d\phi + r^2 \sin\theta\left(1-\sin^2\theta\right) dr \wedge d\theta \wedge d\phi \\ &= r^2 \sin^3\theta\, dr \wedge d\theta \wedge d\phi + r^2 \sin\theta\, dr \wedge d\theta \wedge d\phi - r^2 \sin^3\theta dr \wedge d\theta \wedge d\phi \\ &= r^2 \sin\theta\, dr \wedge d\theta \wedge d\phi \end{aligned}$$

(b) Show $vol\left(D^3\right) = \frac{4}{3}\pi R^3$ by calculating $\int_{D^3} vol_3$.

$$\begin{aligned} Vol\left(D^3\right) &= \int_{D^3} Vol_3 = \int_{D^3} dx \wedge dy \wedge dz = \int_{D^3} r^3 \sin\theta\, dr \wedge d\theta \wedge d\phi \\ &= \int_0^R r^2\, dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi \\ &= \left.\frac{r^3}{3}\right|_0^R \cdot \left. -\cos\theta\right|_0^\pi \cdot \left.\phi\right|_0^{2\pi} \\ &= \frac{R^3}{3} \cdot 2 \cdot 2\pi \\ &= \frac{4}{3}\pi R^3 \end{aligned}$$

(c) Alternatively consider $\beta = xdy \wedge dz$. Use the fact that $\int_{D^3} \partial\beta = \int_{\partial D^3} \beta$ to show $vol\left(D^3\right) = \frac{4}{3}\pi R^3$.

We have that $\int_{D^3} \partial\beta = \int_{\partial D^3} \beta = \int_{\partial D^3} xdy \wedge dz$

We will start by finding dy and dz. Since we are on S^2 and $r = R$, then $dr = 0$.

$$\begin{aligned} dy &= R\cos\phi \sin\theta\, d\phi + R\sin\phi\cos\theta\, d\theta \\ dz &= -R\sin\theta\, d\theta \end{aligned}$$

Now, we'll find $dy \wedge dz$.

$$\begin{aligned} dy \wedge dz &= -dz \wedge dy \\ &= R\sin\theta\, d\theta \wedge \left(R\cos\phi\sin\theta\, d\phi + R\sin\phi\cos\theta\, d\theta\right) \\ &= R^2 \sin^2\theta \cos\phi\, d\theta \wedge d\phi \end{aligned}$$

Next, we'll find $xdy \wedge dz$.

$$xdy \wedge dz = (R\cos\phi\sin\theta)(R^2\sin^2\theta\cos\phi\, d\theta \wedge d\phi)$$
$$= R^3\sin^3\theta\cos^2\phi\, d\theta \wedge d\phi$$

$$\int_{S^2} xdy \wedge dz = \iint R^3\sin^3\theta\cos^2\phi\, d\theta \wedge d\phi$$
$$= R^3 \int_0^{2\pi}\cos^2\phi\, d\phi \int_0^{\pi}\sin^3\theta\, d\theta$$
$$= R^3 \int_0^{2\pi}\frac{1}{2}(1+\cos 2\theta)d\phi \int_0^{\pi}(1-\cos^2\theta)\sin\theta\, d\theta$$
$$= R^3 \int_0^{2\pi}\frac{1}{2}d\phi \int_1^{-1}(u^2-1)du \qquad (let\ u = \cos\theta)$$
$$= R^3 \cdot \pi \cdot \left.\frac{u^3}{3}\right|_1^{-1}$$
$$= \frac{4}{3}\pi R^3$$

(d) Argue by symmetry that $area(D^3) = 4\pi R^2$.

$$vol(D^3) = \frac{4}{3}\pi r^3$$
$$d(vol(D^3)) = d\left(\frac{4}{3}\pi r^3\right)$$
$$d(vol(D^3)) = 4\pi r^2 dr$$
$$\frac{d(vol(D^3))}{dr} = 4\pi r^2$$
$$area(D^3) = 4\pi R^2$$

53. Consider $D^4 = \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 \le R^2\} \subset R^4$. Notice that $\partial D^4 = S^3 = \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 = R^2\}$. The natural parameterization of D^4 is $x = r\cos\phi\sin\theta\sin\psi$, $y = r\sin\phi\sin\theta\sin\psi$, $z = r\cos\theta\sin\psi$ and $w = r\cos\psi$ where $0 \le r \le R$, $0 \le \phi \le 2\pi$, $0 \le \theta \le \pi$ and $0 \le \psi \le \pi$.

(a) Show $dx \wedge dy \wedge dz \wedge dw = r^3 \sin^2\psi \sin\theta\, dr \wedge d\theta \wedge d\phi \wedge d\psi$

$$dx = \frac{dx}{dr}dr + \frac{dx}{d\phi}d\phi + \frac{dx}{d\theta}d\theta + \frac{dx}{d\psi}d\psi$$
$$= \cos\phi\sin\theta\sin\psi\, dr - r\sin\phi\sin\theta\sin\psi\, d\phi + r\cos\phi\cos\theta\sin\psi\, d\theta + r\cos\phi\sin\theta\cos\psi\, d\psi$$

$$dy = \frac{dy}{dr}dr + \frac{dy}{d\phi}d\phi + \frac{dy}{d\theta}d\theta + \frac{dy}{d\psi}d\psi$$
$$= \sin\phi\sin\theta\sin\psi\, dr + r\cos\phi\sin\theta\sin\psi\, d\phi + r\sin\phi\cos\theta\sin\psi\, d\theta + r\sin\phi\sin\theta\cos\psi\, d\psi$$

$$dz = \frac{dz}{dr}dr + \frac{dz}{d\phi}d\phi + \frac{dz}{d\theta}d\theta + \frac{dz}{d\psi}d\psi$$
$$= \cos\theta\sin\psi\, dr - r\sin\theta\sin\psi\, d\theta + r\cos\theta\cos\psi\, d\psi$$

$$dw = \frac{dw}{dr}dr + \frac{dw}{d\phi}d\phi + \frac{dw}{d\theta}d\theta + \frac{dw}{d\psi}d\psi$$
$$= \cos\psi\, dr - r\sin\psi\, d\psi$$

We'll start by finding $dx \wedge dy$

$$\begin{aligned} dx \wedge dy = {} & \cos^2\phi\sin^2\theta\sin^2\psi\, dr \wedge d\phi + r\cos\phi\sin\theta\sin\phi\cos\theta\sin^2\psi\, dr \wedge d\theta \\ & + r\cos\phi\sin\phi\sin^2\theta\sin\psi\cos\psi\, dr \wedge d\psi - r\sin^2\phi\sin^2\theta\sin^2\psi\, d\phi \wedge dr \\ & - r^2\sin^2\phi\sin\theta\cos\theta\sin^2\psi\, d\phi \wedge d\theta - r^2\sin^2\phi\sin^2\theta\sin\psi\cos\psi\, d\phi \wedge d\psi \\ & + r\cos\phi\sin\phi\cos\theta\sin\theta\sin^2\psi\, d\theta \wedge dr + r^2\cos^2\phi\sin\theta\cos\theta\sin^2\psi\, d\theta \wedge d\phi \\ & + r^2\cos\phi\sin\phi\cos\theta\sin\theta\sin\psi\cos\psi\, d\theta \wedge d\psi \\ & + r\cos\phi\sin\phi\sin^2\theta\cos\psi\sin\psi\, d\psi \wedge dr + r^2\cos^2\phi\sin^2\theta\cos\psi\sin\psi\, d\psi \wedge d\phi \\ & + r^2\sin\phi\cos\phi\sin\theta\cos\theta\sin\psi\cos\psi\, d\psi \wedge d\theta \end{aligned}$$

$$= \left(r\cos^2\phi\sin^2\theta\sin^2\psi + r\sin^2\phi\sin^2\theta\sin^2\psi\right)dr\wedge d\phi$$
$$+\left(r\cos\phi\sin\theta\sin\phi\cos\theta\sin^2\psi - r\cos\phi\sin\phi\cos\theta\sin\theta\sin^2\psi\right)dr\wedge d\theta$$
$$+\left(r\cos\phi\sin\phi\sin^2\theta\sin\psi\cos\psi - r\cos\phi\sin\phi\sin^2\theta\cos\psi\sin\psi\right)dr\wedge d\psi$$
$$+\left(r^2\cos^2\phi\sin\theta\cos\theta\sin^2\psi + r^2\sin^2\phi\sin\theta\cos\theta\sin^2\psi\right)d\theta\wedge d\phi$$
$$+\left(r^2\cos\phi\sin\phi\cos\theta\sin\theta\sin\psi\cos\psi - r^2\sin\phi\cos\phi\sin\theta\cos\theta\sin\psi\cos\psi\right)d\theta\wedge d\psi$$
$$+\left(r^2\cos^2\phi\sin^2\theta\cos\psi\sin\psi + r^2\sin^2\phi\sin^2\theta\sin\psi\cos\psi\right)d\psi\wedge d\phi$$

$$= \left(r\sin^2\theta\sin^2\psi\left(\cos^2\phi + \sin^2\phi\right)\right)dr\wedge d\phi$$
$$+\left(r^2\sin\theta\cos\theta\sin^2\psi\left(\cos^2\phi + \sin^2\phi\right)\right)d\theta\wedge d\phi$$
$$+\left(r^2\sin^2\theta\cos\psi\sin\psi\left(\cos^2\phi + \sin^2\phi\right)\right)d\psi\wedge d\phi$$

$$= r\sin^2\theta\sin^2\psi\, dr\wedge d\phi + r^2\sin\theta\cos\theta\sin^2\psi\, d\theta\wedge d\phi + r^2\sin^2\theta\cos\psi\sin\psi\, d\psi\wedge d\phi$$

Next we'll find $dx\wedge dy\wedge dz$.

$$dx\wedge dy\wedge dz = (r\sin^2\theta\sin^2\psi\, dr\wedge d\phi + r^2\sin\theta\cos\theta\sin^2\psi\, d\theta\wedge d\phi +$$
$$r^2\sin^2\theta\cos\psi\sin\psi\, d\psi\wedge d\phi)\wedge\left(\cos\theta\sin\psi\, dr - r\sin\theta\sin\psi\, d\theta + r\cos\theta\cos\psi\, d\psi\right)$$

$$= -r^2\sin^3\theta\sin^3\psi\, dr\wedge d\phi\wedge d\theta + r^2\sin\theta\cos^2\theta\sin^3\psi\, d\theta\wedge d\phi\wedge dr$$
$$+ r^2\cos\theta\sin^2\theta\cos\psi\sin^2\psi\, d\psi\wedge d\phi\wedge dr$$
$$+ r^3\sin\theta\cos^2\theta\sin^2\psi\cos\psi\, d\theta\wedge d\phi\wedge d\psi$$
$$+ r^2\sin^2\theta\cos\theta\cos\psi\sin^2\psi\, d\psi\wedge d\phi\wedge dr$$
$$- r^3\sin^3\theta\sin^2\psi\cos\psi\, d\psi\wedge d\phi\wedge d\theta$$

$$= \left(r^2\sin^3\theta\sin^3\psi + r^2\sin\theta\cos^2\theta\sin^3\psi\right)dr\wedge d\theta\wedge d\phi$$
$$+\left(r^2\cos\theta\sin^2\theta\cos\psi\sin^2\psi - r^2\sin^2\theta\cos\theta\cos\psi\sin^2\psi\right)dr\wedge d\phi\wedge d\psi$$
$$+\left(r^3\sin\theta\cos^2\theta\sin^2\psi\cos\psi + r^3\sin^3\theta\sin^2\psi\cos\psi\right)d\theta\wedge d\phi\wedge d\psi$$

$$= \left(r^2\sin\theta\sin^3\psi\left(\sin^2\theta + \cos^2\theta\right)\right)dr\wedge d\theta\wedge d\phi$$
$$+\left(r^3\sin\theta\sin^2\psi\cos\psi\left(\cos^2\theta + \sin^2\theta\right)\right)d\theta\wedge d\phi\wedge d\psi$$

$$= r^2\sin\theta\sin^3\psi\, dr\wedge d\theta\wedge d\phi + r^3\sin\theta\sin^2\psi\cos\psi\, d\theta\wedge d\phi\wedge d\psi$$

Finally we'll find $dx\wedge dy\wedge dz\wedge dw$.

$$dx\wedge dy\wedge dz\wedge dw = (r^2\sin\theta\sin^3\psi\, dr\wedge d\theta\wedge d\phi$$
$$+ r^3\sin\theta\sin^2\psi\cos\psi\, d\theta\wedge d\phi\wedge d\psi)\wedge\left(\cos\psi\, dr - r\sin\psi\, d\psi\right)$$

$$= -r^3 \sin\theta \sin^4\psi \, dr \wedge d\theta \wedge d\phi \wedge d\psi$$
$$+ r^3 \sin\theta \sin^2\psi \cos^2\psi \, d\theta \wedge d\phi \wedge d\psi \wedge dr$$

$$= r^3 \sin\theta \sin^2\psi \left(\sin^2\psi + \cos^2\psi\right) dr \wedge d\phi \wedge d\theta \wedge d\psi$$

$$= r^3 \sin\theta \sin^2\psi \; dr \wedge d\phi \wedge d\theta \wedge d\psi$$

(b) Show $vol(D^4) = \frac{1}{2}\pi^2 R^4$ by calculating $\int_{D^4} vol_4$.

$$Vol(D^4) = \int_{D^4} vol_4 = \int_{D^4} dx \wedge dy \wedge dz \wedge dw = \int_{D^4} r^3 \sin^2\psi \sin\theta \, dr \wedge d\phi \wedge d\theta \wedge d\psi$$

$$= \int_0^R r^3 dr \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{\pi} \sin^2\psi \, d\psi \qquad \text{[Note: } \sin^2 x = \tfrac{1}{2}(1 - \cos 2x)\text{]}$$

$$= \frac{r^4}{4}\Big|_0^R \cdot \theta\Big|_0^{2\pi} \cdot -\cos\theta\Big|_0^{\pi} \cdot \left(\frac{\psi}{2} - \frac{\sin 2\psi}{4}\right)\Big|_0^{\pi}$$

$$= \frac{R^4}{4} \cdot 2\pi \cdot 2 \cdot \frac{\pi}{2}$$

$$= \frac{1}{2}\pi^2 R^4$$

(c) Alternatively consider $\beta = x dy \wedge dz \wedge dw$. Use the fact that $\int_{D^4} \partial\beta = \int_{\partial D^4} \beta$ to show $vol(D^4) = \frac{1}{2}\pi^2 R^4$.

We have that $\int_{D^3} \partial\beta = \int_{\partial D^3} \beta = \int_{\partial D^3} x dy \wedge dz \wedge dw$

We will start by finding *dy, dz* and *dw*. Since we are on S^2 and $r = R$, then $dr = 0$.

$$dy = R\cos\phi \sin\theta \sin\psi \, d\phi + R\sin\phi \cos\theta \sin\psi \, d\theta + R\sin\phi \sin\theta \cos\psi \, d\psi$$
$$dz = -R\sin\theta \sin\psi \, d\theta + R\cos\theta \cos\psi \, d\psi$$
$$dw = -R\sin\psi \, d\psi$$

Now we'll find $dy \wedge dz$.

$$dy \wedge dz = -dz \wedge dy$$
$$= \left(R\sin\theta \sin\psi \, d\theta - R\cos\theta \cos\psi \, d\psi\right) \wedge$$
$$\left(R\cos\phi \sin\theta \sin\psi \, d\phi + R\sin\phi \cos\theta \sin\psi \, d\theta + R\sin\phi \sin\theta \cos\psi \, d\psi\right)$$

$$= R^2 \cos\phi \sin^2\theta \sin^2\psi \, d\theta \wedge d\phi + R^2 \sin\phi \sin^2\theta \sin\psi \cos\psi \, d\theta \wedge d\psi$$
$$- R^2 \cos\phi \cos\theta \sin\theta \cos\psi \sin\psi \, d\psi \wedge d\phi$$
$$- R^2 \sin\phi \cos^2\theta \sin\psi \cos\psi \, d\psi \wedge d\theta$$

$$= R^2 \cos\phi \sin^2\theta \sin^2\psi \, d\theta \wedge d\phi$$
$$+ \left(R^2 \sin\phi \sin\psi \cos\psi \left(\sin^2\theta + \cos^2\theta\right)\right) d\theta \wedge d\psi$$
$$+ R^2 \cos\phi \cos\theta \sin\theta \cos\psi \sin\psi \, d\phi \wedge d\psi$$

$$= R^2 \cos\phi \sin^2\theta \sin^2\psi \, d\theta \wedge d\phi + R^2 \sin\phi \sin\psi \cos\psi \, d\theta \wedge d\psi$$
$$+ R^2 \cos\phi \cos\theta \sin\theta \cos\psi \sin\psi \, d\phi \wedge d\psi$$

Next we'll find $dy \wedge dz \wedge dw$.

$$dy \wedge dz \wedge dw = (R^2 \cos\phi \sin^2\theta \sin^2\psi \, d\theta \wedge d\phi + R^2 \sin\phi \sin\psi \cos\psi \, d\theta \wedge d\psi$$
$$+ R^2 \cos\phi \cos\theta \sin\theta \cos\psi \sin\psi \, d\phi \wedge d\psi) \wedge \left(-R \sin\psi \, d\psi\right)$$

$$= -R^3 \cos\phi \sin^2\theta \sin^3\psi \, d\theta \wedge d\phi \wedge d\psi$$

$$= R^3 \cos\phi \sin^2\theta \sin^3\psi \, d\phi \wedge d\theta \wedge d\psi$$

Now we'll find $xdy \wedge dz \wedge dw$.

$$xdy \wedge dz \wedge dw = \left(R \cos\phi \sin\theta \sin\psi\right)\left(R^3 \cos\phi \sin^2\theta \sin^3\psi \, d\phi \wedge d\theta \wedge d\psi\right)$$
$$= R^4 \cos^2\phi \sin^3\theta \sin^4\psi \, d\phi \wedge d\theta \wedge d\psi$$

Finally, we'll integrate.

$$\int_{D^4} xdy \wedge dz \wedge dw = \int_{D^4} R^4 \cos^2\phi \sin^3\theta \sin^4\psi \, d\phi \wedge d\theta \wedge d\psi$$
$$= R^4 \int_0^{2\pi} \cos^2\phi \, d\phi \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{\pi} \sin^4\psi \, d\psi$$
$$= R^4 \int_0^{2\pi} \cos^2\phi \, d\phi \int_0^{\pi} (1-\cos^2\theta)\sin\theta \, d\theta \int_0^{\pi} \left[\tfrac{1}{2}(1-\cos 2\psi)\right]^2 d\psi$$
$$= R^4 \int_0^{2\pi} \tfrac{1}{2}(1-\cos(2\phi) \to 0)\, d\phi \int_1^{-1} (u^2-1)\, du \int_0^{\pi} \tfrac{1}{4}(1-2\cos(2\psi)+\cos^2(2\psi))\, d\psi$$
$$= R^4 \int_0^{2\pi} \tfrac{1}{2}\, d\phi \int_1^{-1} (u^2-1)\, du \int_0^{\pi} \tfrac{1}{4}(1-2\cos(2\psi)+\tfrac{1}{2}(1+\cos(4\psi)))\, d\psi$$
$$= R^4 \cdot \pi \cdot \frac{4}{3} \cdot \int_0^{\pi} (\tfrac{3}{8} - 2\cos(2\psi) + \tfrac{1}{8}\cos(4\psi))\, d\psi$$
$$= \frac{4}{3}\pi R^4 \cdot \left(\tfrac{3}{8}\psi - \tfrac{1}{4}\sin(2\psi) \to 0 - \tfrac{1}{32}\sin(4\psi) \to 0\right)\Big|_0^{\pi}$$
$$= \frac{4}{3}\pi R^4 \cdot \frac{3}{8}\pi$$
$$= \frac{1}{2}\pi^2 R^4$$

(d) Argue by symmetry that $area(D^4) = 2\pi^2 R^3$.

$$vol(D^4) = \frac{1}{2}\pi^2 r^4$$
$$d(vol(D^4)) = d\left(\frac{1}{2}\pi^2 r^4\right)$$
$$d(vol(D^4)) = 2\pi^2 r^3 dr$$
$$\frac{d(vol(D^4))}{dr} = 2\pi^2 r^3$$
$$area(D^4) = 2\pi^2 R^3$$

54. Verify my claim that in 5-dimensional Minkowski space we find that $*(dt) = -dx \wedge dy \wedge dz \wedge dw$.

$$\begin{aligned}
*(dt) &= \frac{1}{p!}\frac{1}{(n-p)!}\gamma^i \in_{ijklm} dx^j \wedge dx^k \wedge dx^l \wedge dx^m \\
&= \frac{1}{p!}\frac{1}{(n-p)!}(-1)\delta_i^0 \in_{ijklm} dx^j \wedge dx^k \wedge dx^l \wedge dx^m \\
&= \frac{1}{1!}\frac{-1}{(5-1)!} \in_{0jklm} dx^j \wedge dx^k \wedge dx^l \wedge dx^m \\
&= -\frac{1}{24}(\in_{01234} dx \wedge dy \wedge dz \wedge dw + \in_{01243} dx \wedge dy \wedge dw \wedge dz \\
&\quad + \in_{01324} dx \wedge dz \wedge dy \wedge dw + \in_{01342} dx \wedge dz \wedge dw \wedge dy \\
&\quad + \in_{01423} dx \wedge dw \wedge dy \wedge dz + \in_{01432} dx \wedge dw \wedge dz \wedge dy \\
&\quad + \in_{02134} dy \wedge dx \wedge dz \wedge dw + \in_{02143} dy \wedge dx \wedge dw \wedge dz \\
&\quad + \in_{02314} dy \wedge dz \wedge dx \wedge dw + \in_{02341} dy \wedge dz \wedge dw \wedge dx \\
&\quad + \in_{02413} dy \wedge dw \wedge dx \wedge dz + \in_{02431} dy \wedge dw \wedge dz \wedge dx \\
&\quad + \in_{03124} dz \wedge dx \wedge dy \wedge dw + \in_{03142} dz \wedge dx \wedge dw \wedge dy \\
&\quad + \in_{03214} dz \wedge dy \wedge dx \wedge dw + \in_{03241} dz \wedge dy \wedge dw \wedge dx \\
&\quad + \in_{03412} dz \wedge dw \wedge dx \wedge dy + \in_{03421} dz \wedge dw \wedge dy \wedge dx \\
&\quad + \in_{04123} dw \wedge dx \wedge dy \wedge dz + \in_{04132} dw \wedge dx \wedge dz \wedge dy \\
&\quad + \in_{04213} dw \wedge dy \wedge dx \wedge dz + \in_{04231} dw \wedge dy \wedge dz \wedge dx \\
&\quad + \in_{04312} dw \wedge dz \wedge dx \wedge dy + \in_{04321} dw \wedge dz \wedge dy \wedge dx) \\
&= -\frac{1}{24}(24 \in_{01234} dx \wedge dy \wedge dz \wedge dw) \\
&= -\frac{1}{24}(24(1) dx \wedge dy \wedge dz \wedge dw) \\
&= -dx \wedge dy \wedge dz \wedge dw
\end{aligned}$$