MA 430
Homework #1, part 2
Due Sept. 5, 2006

5. *Prove parts (viii) and (ix) of proposition 3.3.1*

(viii) $\nabla(\vec{F}\bullet\vec{G}) = \vec{F}\times(\nabla\times\vec{G}) + \vec{G}\times(\nabla\times\vec{F}) + (\vec{F}\bullet\nabla)\vec{G} + (\vec{G}\bullet\nabla)\vec{F}$

(1) (2) (3) (4)

- Start by expanding the four terms on the RHS of the equation:

(1) $\vec{F}\times(\nabla\times\vec{G}) = e_k\varepsilon_{ijk}F_i(\nabla\times\vec{G})_j$

$= e_k\varepsilon_{ijk}F_i\varepsilon_{abj}\partial_a G_b$

$= e_k\varepsilon_{ijk}\varepsilon_{abj}F_i(\partial_a G_b)$

$= -e_k\varepsilon_{ikj}\varepsilon_{jab}F_i(\partial_a G_b)$ — Reordered indices: $\varepsilon_{ijk} = -\varepsilon_{ikj}$ and $\varepsilon_{abj} = \varepsilon_{jab}$

$= -e_k(\delta_{ia}\delta_{kb} - \delta_{ka}\delta_{ib})F_i(\partial_a G_b)$ — Substituted $\delta_{ia}\delta_{kb} - \delta_{ka}\delta_{ib}$ for $\varepsilon_{ikj}\varepsilon_{jab}$

$= e_k[\delta_{ka}\delta_{ib}F_i(\partial_a G_b) - \delta_{ia}\delta_{kb}F_i(\partial_a G_b)]$ — Distributed terms

$= e_k[F_i(\partial_k G_i) - F_i(\partial_i G_k)]$ — Substitutions: (1st terms) $k = a$, $i = b$; (2nd terms) $i = a$, $k = b$

(2) $\vec{G}\times(\nabla\times\vec{F}) = e_k\varepsilon_{ijk}G_i(\nabla\times\vec{F})_j$

$= e_k\varepsilon_{ijk}G_i\varepsilon_{abj}\partial_a F_b$

$= e_k\varepsilon_{ijk}\varepsilon_{abj}G_i(\partial_a F_b)$

$= -e_k\varepsilon_{ikj}\varepsilon_{jab}G_i(\partial_a F_b)$ — Reordered indices: $\varepsilon_{ijk} = -\varepsilon_{ikj}$ and $\varepsilon_{abj} = \varepsilon_{jab}$

$= -e_k(\delta_{ia}\delta_{kb} - \delta_{ka}\delta_{ib})G_i(\partial_a F_b)$ — Substituted $\delta_{ia}\delta_{kb} - \delta_{ka}\delta_{ib}$ for $\varepsilon_{ikj}\varepsilon_{jab}$

$= e_k[\delta_{ka}\delta_{ib}G_i(\partial_a F_b) - \delta_{ia}\delta_{kb}G_i(\partial_a F_b)]$ — Distributed terms

$= e_k[G_i(\partial_k F_i) - G_i(\partial_i F_k)]$ — Substitutions: (1st terms) $k = a$, $i = b$; (2nd terms) $i = a$, $k = b$

(3) $(\vec{F}\bullet\nabla)\vec{G} = F_i\partial_i(G_k e_k) = e_k F_i(\partial_i G_k)$

(4) $(\vec{G}\bullet\nabla)\vec{F} = G_i\partial_i(F_k e_k) = e_k G_i(\partial_i F_k)$

- Plug in new terms:

$$\vec{F}\times(\nabla\times\vec{G}) + \vec{G}\times(\nabla\times\vec{F}) + (\vec{F}\bullet\nabla)\vec{G} + (\vec{G}\bullet\nabla)\vec{F} = e_k[F_i(\partial_k G_i) - F_i(\partial_i G_k) + G_i(\partial_k F_i) - G_i(\partial_i F_k) + F_i(\partial_i G_k) + G_i(\partial_i F_k)]$$

$= e_k[F_i(\partial_k G_i) + G_i(\partial_k F_i)]$ — Simplified terms that canceled out

$= e_k[\partial_k(F_i G_i)]$

$= e_k[\partial_k(\vec{F}\bullet\vec{G})]$

$= \nabla(\vec{F}\bullet\vec{G})$

(ix) $\nabla\times\left(\vec{F}\times\vec{G}\right)=\left(\vec{G}\bullet\nabla\right)\vec{F}-\left(\vec{F}\bullet\nabla\right)\vec{G}+\vec{F}\left(\nabla\bullet\vec{G}\right)-\vec{G}\left(\nabla\bullet\vec{F}\right)$

- Start by expanding the terms on the RHS of the equation:

$$\begin{aligned}\left(\vec{G}\bullet\nabla\right)\vec{F}-\left(\vec{F}\bullet\nabla\right)\vec{G}+\vec{F}\left(\nabla\bullet\vec{G}\right)-\vec{G}\left(\nabla\bullet\vec{F}\right) &= G_k\partial_k\left(F_m e_m\right)-F_k\partial_k\left(G_m e_m\right)+F_m e_m\left(\partial_k G_k\right)-G_m e_m\left(\partial_k F_k\right)\\ &= e_m\left[G_k\left(\partial_k F_m\right)-F_k\left(\partial_k G_m\right)+F_m\left(\partial_k G_k\right)-G_m\left(\partial_k F_k\right)\right]\end{aligned}$$

- Next, expand the terms on the LHS of the equation:

$$\nabla\times\left(\vec{F}\times\vec{G}\right)=e_m\varepsilon_{ijm}\partial_i\left(\vec{F}\times\vec{G}\right)_j$$

$$=e_m\varepsilon_{ijm}\partial_i\left(\varepsilon_{abj}F_aG_b\right)$$

$$=e_m\varepsilon_{ijm}\varepsilon_{abj}\partial_i\left(F_aG_b\right)$$

$$=e_m\varepsilon_{ijm}\varepsilon_{abj}\left[\left(\partial_iF_a\right)G_b+F_a\left(\partial_iG_b\right)\right]$$ Expanded partial derivatives

$$=-e_m\varepsilon_{imj}\varepsilon_{jab}\left[\left(\partial_iF_a\right)G_b+F_a\left(\partial_iG_b\right)\right]$$ Reordered indices: $\varepsilon_{ijm}=-\varepsilon_{imj}$ and $\varepsilon_{abj}=\varepsilon_{jab}$

$$=-e_m\left(\delta_{ia}\delta_{mb}-\delta_{ma}\delta_{ib}\right)\left[\left(\partial_iF_a\right)G_b+F_a\left(\partial_iG_b\right)\right]$$ Substituted $\delta_{ia}\delta_{kb}-\delta_{ka}\delta_{ib}$ for $\varepsilon_{ikj}\varepsilon_{jab}$

$$=e_m\left(\delta_{ma}\delta_{ib}-\delta_{ia}\delta_{mb}\right)\left[\left(\partial_iF_a\right)G_b+F_a\left(\partial_iG_b\right)\right]$$ Distributed negative

$$=e_m\left[\delta_{ma}\delta_{ib}\left(\partial_iF_a\right)G_b+\delta_{ma}\delta_{ib}F_a\left(\partial_iG_b\right)-\delta_{ia}\delta_{mb}\left(\partial_iF_a\right)G_b-\delta_{ia}\delta_{mb}F_a\left(\partial_iG_b\right)\right]$$ Expanded terms

$$=e_m\left[\left(\partial_iF_m\right)G_i+F_m\left(\partial_iG_i\right)-\left(\partial_iF_i\right)G_m-F_i\left(\partial_iG_m\right)\right]$$ Substitutions: (1st) $m=a,\ i=b$, (2nd) $i=a,\ m=b$

$$=e_m\left[G_i\left(\partial_iF_m\right)+F_m\left(\partial_iG_i\right)-G_m\left(\partial_iF_i\right)-F_i\left(\partial_iG_m\right)\right]$$ Rearranged the order of the terms

$$=e_m\left[G_i\left(\partial_iF_m\right)-G_m\left(\partial_iF_i\right)+F_m\left(\partial_iG_i\right)-F_i\left(\partial_iG_m\right)\right]$$ Reordered terms again to match RHS terms

Therefore, $\nabla\times\left(\vec{F}\times\vec{G}\right)=\left(\vec{G}\bullet\nabla\right)\vec{F}-\left(\vec{F}\bullet\nabla\right)\vec{G}+\vec{F}\left(\nabla\bullet\vec{G}\right)-\vec{G}\left(\nabla\bullet\vec{F}\right)$.

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6. *Prove proposition 3.3.2*

- Divergence of the curl: $\nabla\bullet\left(\nabla\times\vec{F}\right)=0$

$$\nabla\bullet\left(\nabla\times\vec{F}\right)=\partial_k\left[\left(\nabla\times\vec{F}\right)\right]_k$$

$$=\partial_k\varepsilon_{ijk}\partial_iF_j$$

$$=\varepsilon_{ijk}\partial_k\partial_iF_j$$ Rearranged order of delta terms

$$=\varepsilon_{mjp}\partial_p\partial_mF_j$$ Substitutions for variables: $i=m,\ k=p$

$$=\varepsilon_{mjp}\partial_m\partial_pF_j$$ Interchange order of delta terms ← $\left(\frac{\partial}{\partial x^p}\frac{\partial}{\partial x^m}=\frac{\partial}{\partial x^m}\frac{\partial}{\partial x^p}\quad \partial_p\partial_m=\partial_m\partial_p\right)$

$$=-\varepsilon_{pjm}\partial_m\partial_pF_j$$ Reordered indices: $\varepsilon_{mjp}=-\varepsilon_{pjm}$

$$=-\partial_m\varepsilon_{pjm}\partial_pF_j$$ Rearranged order of delta terms

$$=-\partial_m\left[\left(\nabla\times\vec{F}\right)\right]_m$$

$$=-\nabla\bullet\left(\nabla\times\vec{F}\right)$$

$$2\left[\nabla\bullet\left(\nabla\times\vec{F}\right)\right]=0$$

$$\nabla\bullet\left(\nabla\times\vec{F}\right)=0$$

- Curl of the gradient: $\nabla \times (\nabla f) = 0$

$$
\begin{aligned}
\nabla \times (\nabla f) &= e_k \varepsilon_{ijk} \partial_i (\nabla f)_j \\
&= e_k \varepsilon_{ijk} \partial_i \partial_j f \\
&= e_k \varepsilon_{mpk} \partial_m \partial_p f && \text{Substitutions for variables: } i = m,\ j = p \\
&= e_k \varepsilon_{mpk} \partial_p \partial_m f && \text{Interchange order of delta terms} \\
&= -e_k \varepsilon_{pmk} \partial_p \partial_m f && \text{Reordered indices: } \varepsilon_{mpk} = -\varepsilon_{pmk} \\
&= -e_k \varepsilon_{pmk} \partial_p (\nabla f) \\
&= -\nabla \times (\nabla f) \\
2[\nabla \times (\nabla f)] &= 0 \\
\nabla \times (\nabla f) &= 0
\end{aligned}
$$

- Curl of the curl: $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \bullet \vec{F}) - \nabla^2 \vec{F}$

$$
\begin{aligned}
\nabla \times (\nabla \times \vec{F}) &= e_k \varepsilon_{ijk} \partial_i (\nabla \times \vec{F})_j \\
&= e_k \varepsilon_{ijk} \partial_i \varepsilon_{abj} \partial_a F_b \\
&= e_k \varepsilon_{ijk} \partial_i \varepsilon_{abj} \partial_a F_b \\
&= e_k \varepsilon_{ijk} \varepsilon_{abj} \partial_i \partial_a F_b && \text{Rearranged order of delta and epsilon terms} \\
&= -e_k \varepsilon_{ikj} \varepsilon_{jab} \partial_i \partial_a F_b && \text{Reordered indices: } \varepsilon_{ijk} = -\varepsilon_{ikj} \text{ and } \varepsilon_{abj} = \varepsilon_{jab} \\
&= -e_k (\delta_{ia}\delta_{kb} - \delta_{ka}\delta_{ib}) \partial_i \partial_a F_b && \text{Substituted } \delta_{ia}\delta_{kb} - \delta_{ka}\delta_{ib} \text{ for } \varepsilon_{ikj}\varepsilon_{jab} \\
&= e_k [\delta_{ka}\delta_{ib} \partial_i \partial_a F_b - \delta_{ia}\delta_{kb} \partial_i \partial_a F_b] && \text{Distributed terms} \\
&= e_k [\partial_i \partial_k F_i - \partial_i \partial_i F_k] && \text{Substitutions: (1}^{\text{st}}\text{) } k = a,\ i = b,\ \text{(2}^{\text{nd}}\text{) } i = a,\ k = b \\
&= e_k [\partial_k \partial_i F_i - \partial_i \partial_i F_k] && \text{Reordered first pair of delta terms} \\
&= e_k [\partial_k (\nabla \bullet \vec{F}) - \nabla^2 F_k] \\
&= \nabla(\nabla \bullet \vec{F}) - \nabla^2 \vec{F}
\end{aligned}
$$