Ma 430 : PROBLEMS ABOUT EUCLIDEAN STRUCTURES & ROTATIONS & MORE

PROBLEM 12 We defined $O(3) = \{A \in \mathbb{R}^{3\times3} \mid A^TA = I\}$ prove the following

a.) $O(3)$ is a group with operation matrix multiplication. You may assume matrix multiplication is associative.

b.) $SO(3) = \{A \in O(3) \mid \det(A) = 1\}$. Show that $SO(3)$ is a group.

c.) Let $\mathcal{e} = -I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ then define the coset of $SO(3)$

$$SO(3)\mathcal{e} \equiv \{B \in \mathbb{R}^{3\times3} \mid \exists A \in SO(3) \text{ such that } B = A\mathcal{e}\}$$

Show that $O(3) = SO(3) \cup SO(3)\mathcal{e}$ and $SO(3) \cap SO(3)\mathcal{e} = \phi$.

PROBLEM 13 Define $\mathcal{R} = \{\varphi \mid \varphi: \mathbb{R}^3 \to \mathbb{R}^3 \text{ is a rigid motion}\}$. Show that $\mathcal{R}$ is a group with respect to the operation of function composition. You may assume that function composition is associative.

PROBLEM 14 Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any 2×2 real matrix. Assume that $\det(A) = 1$. Show that A is orthogonal iff $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Then show that this is the case iff there is some angle θ such that $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. This problem shows every special orthogonal matrix is a rotation in 2-dim.

PROBLEM 15 Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x_1, x_2, x_3) = \left(\tfrac{\sqrt{3}}{2}x_1 + \tfrac{1}{2}x_3,\ x_2,\ -\tfrac{1}{2}x_1 + \tfrac{\sqrt{3}}{2}x_3\right)$$

(a.) use the definition of an orthogonal transformation to show that f is orthogonal

(b.) Find the matrix A of f (meaning find A such that $f(x) = Ax$) then show by matrix multiplication that $A^TA = I$.

(c.) find an angle θ such that

$$A = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

PROBLEM 16

Assume that the surface of the earth is represented by the (xy)-plane and that the acceleration of gravity is $-gk$. If a projectile of mass m is launched from the origin with initial speed v_o at an angle α with the horizontal then find

(a) the position of the projectile as a function of t

(b) the time it takes to attain maximum height

(c) the range of the projectile

(d) Show that the angle α which gives maximum range is $\alpha = 45°$.

Assume throughout the problem that the initial velocity vector lies in the (yz)-plane.

PROBLEM 17

~~2.~~ A train moves due north at a constant speed v. It passes a carousel one mile due west of the carousel at 1:00 p.m. Consider two coordinate systems with origin the central point about which the carousel rotates. The first system, $x = (x_1, x_2)$ is oriented in the usual East-West, North-South fashion, but the second system is attached to the carousel which is rotating counterclockwise with constant angular speed ω with respect to x. The rotating system is denoted by $y_t = (y_{t1}, y_{t2})$ and at 1:00 p.m. $t = 0$ and $y_t = y_0 = x$.

(a) Find the position of the train as a function of t as measured by x.

(b) Find the position of the train as a function of t as observed by y.

(c) Find the velocity of the train in both systems.

(d) At 1:00 pm a car is located at (nearly!) the same point as the train and is traveling northeast at a constant speed u. Find the equation of motion of the car in both systems.

(e) If the train carries a coordinate system $z_t = (z_{t1}, z_{t2})$ which is oriented like x and which lies one mile due west of the origin of x at $t = 0$ find the equation of motion of a point on the rim of the carousel relative to z. To be explicit assume the point on the rim of the carousel has y-coordinates $(a, 0)$ for all t.