PROBLEM 49 The n-form $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n = \mathrm{vol}_n$ is the "volume form" on $\mathbb{R}^n$. In fact if $M \subset \mathbb{R}^n$ is some finite region then $\int_M \mathrm{vol}_n = \mathrm{vol}(M)$. Using the Generalized Stoke's Th$^{\underline{m}}$ show that

$$\mathrm{Vol}(M) = \int_{\partial M} x^1 dx^2 \wedge dx^3 \wedge \cdots \wedge dx^n$$

PROBLEM 50 Explain why $\int_M f\, dx \wedge dy \wedge dz = \iiint_{D_3} f\, dx\, dy\, dz$, where $\Phi : D_3 \to M$ is a parametrization of M by the Cartesian coordinates x, y, z. (think $\Phi(x,y,z) = (x,y,z)$ or $u_1 = x$, $u_2 = y$, $u_3 = z$ in the langauge of def$^{\underline{n}}$ 12.1.6).

PROBLEM 51 Consider $D^2 \subset \mathbb{R}^2$ a dish of radius R. That is $D^2 = \{(x,y) \mid x^2+y^2 \leq R\}$. Note $\partial D^2 = S^1 = \{(x,y) \mid x^2+y^2 = R^2\}$. The natural parametrization of D^2 is $x = r\cos\varphi$, $y = r\sin\varphi$

a.) Show $dx \wedge dy = r\, dr \wedge d\varphi$

b.) Show $\mathrm{vol}(D^2) = \pi r^2$ by calculating $\int_{D^2} \mathrm{vol}_2$.

c.) Alternatively consider $\beta = x\,dy$. Use the fact $\int_{D^2} d\beta = \int_{\partial D^2} \beta$ to show $\mathrm{vol}(D^2) = \pi r^2$.

d.) Argue by symmetry that $\mathrm{area}(D^2) = 2\pi r$.

$\boxed{\text{PROBLEM 52}}$ Consider $D^3 = \{(x,y,z) \mid x^2+y^2+z^2 \le R^2\} \subset \mathbb{R}^3$. Notice that $\partial D^3 = S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = R^2\}$. The natural parametrization of D^3 is $x = r\cos\varphi \sin\theta$, $y = r\sin\varphi\sin\theta$, $z = r\cos\theta$ where $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le \theta \le \pi$.

a.) Show $dx \wedge dy \wedge dz = r^2 \sin\theta \, dr \wedge d\theta \wedge d\varphi$

b.) Show $\text{vol}(D^3) = \frac{4}{3}\pi R^3$ by calculating $\int_{D^3} \text{vol}_3$

c.) Alternatively consider $\beta = x\, dy \wedge dz$. Use the fact $\int_{D^3} d\beta = \int_{\partial D^3} \beta$ to show $\text{vol}(D^3) = \frac{4}{3}\pi R^3$

d.) Argue by symmetry that $\text{area}(D^3) = 4\pi R^2$

$\boxed{\text{PROBLEM 53}}$ Consider $D^4 = \{(x,y,z,w) \mid x^2+y^2+z^2+w^2 \le R^2\} \subset \mathbb{R}^4$. Notice $\partial D^4 = S^3 = \{(x,y,z,w) \mid x^2+y^2+z^2+w^2 = R^2\}$. The natural parametrization is $x = r\cos\varphi\sin\theta\sin\psi$, $y = r\sin\varphi\sin\theta\sin\psi$, $z = r\cos\theta\sin\psi$ and $w = r\cos\psi$ where $0 \le r \le R$, $0 \le \varphi \le 2\pi$ and $0 \le \theta \le \pi$ and $0 \le \psi \le \pi$.

a.) Show $dx \wedge dy \wedge dz \wedge dw = r^3 \sin^2\psi \sin\theta \, dr \wedge d\varphi \wedge d\theta \wedge d\psi$

b.) Show $\text{vol}(D^4) = \frac{1}{2}\pi^2 R^4$ by calculating $\int_{D^4} \text{vol}_4$.

c.) Alternatively consider $\beta = x\, dy \wedge dz \wedge dw$. Use the fact that $\int_{D^4} d\beta = \int_{\partial D^4} \beta$ to show $\text{vol}(D^4) = \frac{1}{2}\pi^2 R^4$

d.) Argue by symmetry that $\text{area}(D^4) = 2\pi^2 R^3$.

$\boxed{\text{PROBLEM 54}}$ Verify my claim that in 5-dim'l Minkowski Space (described in § 12.3) we find that

$$\boxed{*(dt) = -dx \wedge dy \wedge dz \wedge dw}$$