## Mathematical Models in Physics, Final Exam, December 12, 2006

Do all of your work on your own paper, please leave white space between problems. Notice that in each problem I have used **bold** to draw your attention to what I want you to show. Relax, it only looks long. Many of these problems are very short, make sure to use what you've already done when you can.

### Problem 1

(**5pts**) We can define $det(A)$ implicitly by $A^i_1 e_i \wedge A^j_2 e_j = det(A) e_1 \wedge e_2$. Given that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

**Show** using the definition given above that,

$$\boxed{det(A) = ad - bc}$$

### Problem 2

(**5pts**) Recall that if $\alpha$ is a $p$-form

$$\alpha = \frac{1}{p!}\alpha_{i_1 i_2 \dots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}$$

then the Hodge dual is defined by

$$*\alpha \equiv \frac{1}{p!}\frac{1}{(n-p)!}\alpha^{i_1 i_2 \dots i_p} \epsilon_{i_1 i_2 \dots i_p j_1 j_2 \dots j_{n-p}} dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_{n-p}}$$

Consider $\mathbb{R}^3$ with the Euclidean metric, **show** that,

$$\boxed{*dz = dx \wedge dy}$$

### Problem 3

(**5pts**) Given vector fields $A = (a, b, c)$ and $B = (f, g, h)$ recall the work-form mapping, $\omega_A = adx + bdy + cdz$ and the flux-form mapping $\Phi_A = ady \wedge dz + bdz \wedge dx + cdx \wedge dy$. **Show**

$$\boxed{\omega_A \wedge \omega_B = \Phi_{A \times B}}$$

where $A \times B$ denotes the cross-product of $A$ with $B$.

## Problem 4

(**20pts**) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function and let $F = (F_1, F_2, F_3)$ and $G = (G_1, G_2, G_3)$ be smooth vector fields on $\mathbb{R}^3$. You may assume the identities

$$dx^i \wedge dx^j \wedge dx^k = \epsilon_{ijk} dx \wedge dy \wedge dz \qquad \epsilon_{ijk}\epsilon_{mjk} = 2\delta_{im}$$

these are useful for $(ii.)$ below. Also you might save some writing using the more efficient statements of the work and flux form mappings,

$$\omega_F = F_i dx^i \qquad \Phi_G = \frac{1}{2}\epsilon_{ijk} G_i dx^j \wedge dx^k$$

**Show**

(i.) $df = \omega_{\nabla f}$

(ii.) $d\Phi_G = (\nabla \cdot G) dx \wedge dy \wedge dz$

(iii.) $d\omega_F = \Phi_{\nabla \times F}$

(iv.) $d(d\alpha) = 0$ for any $p$-form $\alpha$

## Problem 5

(**10pts**) Given the Generalized Stoke's Theorem $\int_M d\alpha = \int_{\partial M} \alpha$ plus the defintion of differential form integration ( which you have memorized) it can be shown,

$$\int_C \omega_F = \int_C F \cdot dl \qquad \int_S \Phi_G = \int_S G \cdot dA \qquad \int_V f dx \wedge dy \wedge dz = \int\int\int_V f dx dy dz$$

Using the items given above **prove the following statements**

(i.) $\int_S (\nabla \times F) \cdot dA = \int_{\partial S} F \cdot dl$

(ii.) $\int\int\int_V (\nabla \cdot G) dx dy dz = \int_{\partial V} G \cdot dA$

for a curve $C$, a surface $S$ and a volume $V$ all consistently oriented subsets of $\mathbb{R}^3$ and we assume that the vector fields are smooth so all the derivatives listed above are well-behaved.

## Problem 6

(**5pts**) Let us parametrize Minkowski space with $(t, x, y, z) = (x^\mu)$, $\mu = 0, 1, 2, 3$ as usual and the standard metric for our course which has $diag(\eta) = (-1, 1, 1, 1)$. **Show that**,

$$\boxed{*dt = -dx \wedge dy \wedge dz}$$

The following table is for future use on problems **beyond** this point on the test,

| | |
|---|---|
| $^*1 = dt \wedge dx \wedge dy \wedge dz$ | $^*(dt \wedge dx \wedge dy \wedge dz) = -1$ |
| $^*(dx \wedge dy \wedge dz) = -dt$ | $^*dt = -dx \wedge dy \wedge dz$ |
| $^*(dt \wedge dy \wedge dz) = -dx$ | $^*dx = -dy \wedge dz \wedge dt$ |
| $^*(dt \wedge dz \wedge dx) = -dy$ | $^*dy = -dz \wedge dx \wedge dt$ |
| $^*(dt \wedge dx \wedge dy) = -dz$ | $^*dz = -dx \wedge dy \wedge dt$ |
| $^*(dz \wedge dt) = dx \wedge dy$ | $^*(dx \wedge dy) = -dz \wedge dt$ |
| $^*(dx \wedge dt) = dy \wedge dz$ | $^*(dy \wedge dz) = -dx \wedge dt$ |
| $^*(dy \wedge dt) = dz \wedge dx$ | $^*(dz \wedge dx) = -dy \wedge dt$ |

**Problem 7**

(**10pts**) Define $\mathcal{J} = -\rho dt + J_1 dx + J_2 dy + J_3 dz$. **Show**

(i.) $*\mathcal{J} = \rho dx \wedge dy \wedge dz - \Phi_J \wedge dt$
(ii.) $d(*F) = *\mathcal{J} \implies d * \mathcal{J} = 0$.

please use the table to do $(i.)$.

**Problem 8**

( *and now for something completely different...*)
(**10pts**) For an arbitrary $p$-form $\alpha$ and $q$-form $\beta$ is it **true or false** that

(i.) $d(*\alpha) = *(d\alpha)$
(ii.) $*(\alpha \wedge \beta) = *\alpha \wedge *\beta$
(iii.) $\alpha \wedge \beta = -\beta \wedge \alpha$

explain your reasoning, **if false give a counter-example**.

**Problem 9**

(**5pts**) Given that $F = \omega_E \wedge dt + \Phi_B$ use the table to show that,

$$\boxed{*F = -\omega_B \wedge dt + \Phi_E}$$

**Problem 10**

(**25pts**) Recall that Maxwell's equations are,

$$\begin{array}{ll} \text{Gauss Law} & \nabla \cdot E = \rho/\epsilon_o \\ \text{Ampere's Law} & \nabla \times B = \mu_o J + \mu_o \epsilon_o \frac{\partial E}{\partial t} \\ \text{Faradays Law} & \nabla \times E = -\frac{\partial B}{\partial t} \\ \text{no magnetic monopoles} & \nabla \cdot B = 0 \end{array} \tag{1}$$

**Show** that Maxwell's equations are the same as the following differential form equations,

$$\boxed{dF = 0 \qquad\qquad d(*F) = \mu_o * \mathcal{J}}$$

**Bonus I:** What color is Superman's underwear ?

**Bonus II:** Suppose that the one-form potential is

$$\boxed{A = \frac{1}{r(z-r)}(xdy - ydx).}$$

Show that,

$$\boxed{A = -(1+\cos(\theta))d\phi}$$

where $(r, \theta, \phi)$ are spherical coordinates

$$x = r\cos(\phi)\sin(\theta) \qquad y = r\sin(\phi)\sin(\theta) \qquad z = r\cos(\theta).$$

Calculate $dA$, would this satisfy a sort of generalized Maxwell's equations ? What sort of field does this potential represent ? Is this potential only singular where the field is singular ( which is typical of where the actual charge resides, you may assume that the charge resides where the fields are singular think about the Coulomb field for example it is only infinite at the origin where the charge is ), or is the potential also singular at regions where there is no charge ? What effect does the gauge transformation $A' = A + d\phi$ have on the singularity in the potential ?. (notice $\phi$ itself is ill-defined on z-axis so this in a "singular gauge transformation"). You can show

$$\boxed{A' = (1-\cos(\theta))d\phi}$$

Argue that the singularity in $A$ moved somewhere else in $A'$. The ultimate outcome of these observations is we cannot define global gauge potentials for the type of field encountered here, in fact one needs the construction of the principle fiber bundle to give a good mathematical account of what is going on.