**Mathematical Models in Physics, Exam I, October 3, 2006**

Do all of your work on your own paper, please leave white space between problems. Notice that in each problem I have used **bold** to draw your attention to what I want you to show.

### Problem 1

(12pts) **Define** the following,
(a.) rigid motion on $\mathbb{R}^3$
(b.) Euclidean structure on a set $S$
(c.) observer on $S$
(d.) inertial observer on $S$
(e.) Minkowski metric on $\mathbb{R}^4$
(f.) Spacelike, lightlike, and timelike vectors in $\mathbb{R}^4$

### Problem 2

(15 pts) Recall that $\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{jm}\delta_{in}$. Let $E = E_i e_i$ be a vector field that has smooth components, and let $\nabla = e_i \partial_i$ so that we denote,

$$\nabla \cdot E = \partial_i E_i \qquad \nabla \times E = e_k \epsilon_{ijk} \partial_i E_j \qquad \nabla^2 E = \partial_i \partial_i E \qquad \text{where} \quad \partial_i = \frac{\partial}{\partial x^i}$$

**Show** that the following identity holds

$$\boxed{\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E}$$

### Problem 3

(15 pts) Maxwell's equations in free space in the absence of source charges and currents are,

$$\begin{array}{ll} \nabla \cdot E = 0 & \nabla \cdot B = 0 \\ \nabla \times E = -\partial_t B & \nabla \times B = \mu_o \epsilon_o \partial_t E \end{array} \tag{1}$$

where $\mu_o, \epsilon_o$ are constants. **Show that,**

$$\boxed{\nabla^2 E = \mu_o \epsilon_o \frac{\partial^2 E}{\partial t^2}}$$

### Problem 4

(20 pts) Suppose that we have an inertial system of coordinates $(t, x)$ and that there is another frame of reference $(\bar{t}, \bar{x})$ that moves with constant speed $v$ in positive x-direction. Recall that we argued that due to Einstein's first postulate we can relate the coordinates of the two frames of reference as follows,

$$\begin{aligned} \bar{x} &= a(x - vt) \\ x &= a(\bar{x} + v\bar{t}) \end{aligned} \tag{2}$$

where $a$ is a yet undetermined constant. As usual let $c = 1$ so that $\beta = v$. Consider the trajectory of a light beam in both systems. Use the notation $\phi(t)$ for the position of the light according to $(t, x)$

and $\psi(\bar{t})$ for the position of the light according to $(\bar{t}, \bar{x})$. We can relate the trajectories according to what we learned from the 1st postulate,

$$\phi(t) = a(\psi(\bar{t}) + v\bar{t}) \qquad \psi(\bar{t}) = a(\phi(t) - vt)$$

By Einstein's 2nd postulate we know that the speed of light is same in both frames,

$$\frac{d}{dt}(\phi(t)) = 1 \qquad \frac{d}{d\bar{t}}\psi(\bar{t}) = 1.$$

By calculus we know that,

$$\frac{d}{dt}[\psi(\bar{t})] = \frac{d\psi}{d\bar{t}}\frac{d\bar{t}}{dt} \qquad \frac{d}{d\bar{t}}[\phi(t)] = \frac{d\phi}{dt}\frac{dt}{d\bar{t}} \qquad \frac{d\bar{t}}{dt}\frac{dt}{d\bar{t}} = 1$$

Use what is given above to **derive the following equations,**

$$\begin{aligned} \bar{x} &= \gamma(x - \beta t) \\ \bar{t} &= \gamma(t - \beta x). \end{aligned} \tag{3}$$

where $\gamma = 1/\sqrt{1-\beta^2}$.

**Problem 5**

(20 pts) Let $(S, \mathcal{E})$ be a Newtonian space. Let $\mathcal{X}, \mathcal{Y} : I \to \mathcal{E}$ be inertial observers then we proved that they are related by

$$\mathcal{Y} = A\mathcal{X} + tv + w$$

for fixed $A \in SO(3)$, and $v, w \in \mathbb{R}^3$. Consider some particle with trajectory $\gamma : I \to S$. **Show** that if the acceleration of $\gamma$ with respect to $\mathcal{X}$ is zero then the acceleration of $\gamma$ with respect to $\mathcal{Y}$ is likewise zero; that is show $a_{\mathcal{X}} = 0 \implies a_{\mathcal{Y}} = 0$.

**Problem 6**

(10pts) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal transformation, furthermore let $A \in \mathbb{R}^{3\times 3}$ be its matrix; $L(x) = Ax$. **Show that** $A^TA = I$.

**Problem 7**

(10pts) Let $L : \mathbb{R}^4 \to \mathbb{R}^4$ be a Lorentz transformation, furthermore let $B \in \mathbb{R}^{4\times 4}$ be its matrix; $L(x) = Bx$. Recall we denoted the matrix of the Minkowski metric by $\eta$ so that $< x, y >= x^T\eta y$ for all $x, y \in \mathbb{R}^4$. **Show that** $B^T\eta B = \eta$.

**Bonus:** if we define $SO(1,3) = \{B \in \mathbb{R}^{4\times 4} \mid B^T\eta B = \eta\}$ then show that this a group and that there is a homomorphism of $SO(3)$ into this group.

Oct. 3, 2006

1. DEFINE the following:

(a) Rigid motion on $\mathbb{R}^3$ - a function $\phi:\mathbb{R}^3 \to \mathbb{R}^3$ is called a rigid motion if $\exists$ a vector $r \in \mathbb{R}^3$ and a rotation matrix $A \in SO(3)$ such that $\phi(x) = Ax + r$. ✓

(b) Euclidean Structure on a set $S$ - $\mathcal{E}$ is a Euclidean Structure on a set $S$ iff $\mathcal{E}$ is a family of bijections from $S$ onto $\mathbb{R}^3$ such that

(1) $\mathcal{X}, \mathcal{Y} \in \mathcal{E}$, then $\mathcal{X} \circ \mathcal{Y}^{-1}$ is a rigid motion

(2) $\mathcal{X} \in \mathcal{E}$ and $\phi$ is a rigid motion, then $\phi \circ \mathcal{X} \in \mathcal{E}$. ✓

(c) Observer on $S$ - is a function on the interval $I \subset \mathbb{R}$ into $\mathcal{E}$. Such a function $\mathcal{X}: I \to \mathcal{E}$ is a time varying coordinate system on $S$. ✓

(d) Inertial observer on $S$ - An observer $\mathcal{X}: I \to \mathcal{E}$ is said to be an inertial observer iff $\exists$ $\mathcal{X}_0 \in \mathcal{E}$, $A \in SO(3)$, $v, w \in \mathbb{R}^3$ such that $\mathcal{X}_t = A\mathcal{X}_0 + tv + w$ $\forall t \in I$. ✓

(e) Minkowski Metric on $\mathbb{R}^4$ - Let $v = (v^0, v^1, v^2, v^3)^t$, $w = (w^0, w^1, w^2, w^3)^t \in \mathbb{R}^4$, then the Minkowski metric $\langle , \rangle : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ is defined by

$$\langle v, w \rangle = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3$$ ✓

(f) <u>Spacelike, lightlike and timelike vectors in $\mathbb{R}^4$</u> –

Let $v = (v^0, v^1, v^2, v^3) \in \mathbb{R}^4$, then $v$ is a:

- timelike vector if $\langle v, v \rangle < 0$
- lightlike vector if $\langle v, v \rangle = 0$ ✓
- spacelike vector if $\langle v, v \rangle > 0$.

2. Show that $\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E$

$$\nabla \times (\nabla \times E) = e_k \epsilon_{ijk} \partial_i (\nabla \times E)_j$$
$$= e_k \epsilon_{ijk} \partial_i \epsilon_{abj} \partial_a E_b$$
$$= e_k \epsilon_{ijk} \epsilon_{abj} \partial_i \partial_a E_b$$
$$= -e_k \epsilon_{ikj} \epsilon_{abj} \partial_i \partial_a E_b \quad \text{since } \epsilon_{ijk} = -\epsilon_{ikj}$$
$$= -e_k \epsilon_{ikj} \epsilon_{jab} \partial_i \partial_a E_b \quad \text{since } \epsilon_{abj} = \epsilon_{jab}$$
$$= -e_k (\delta_{ia}\delta_{kb} - \delta_{ka}\delta_{ib}) \partial_i \partial_a E_b$$
$$= e_k (\delta_{ka}\delta_{ib} \partial_i \partial_a E_b - \delta_{ia}\delta_{kb} \partial_i \partial_a E_b)$$
$$= e_k (\partial_i \partial_k E_i - \partial_i \partial_i E_k)$$
$$= e_k (\partial_k \partial_i E_i - \partial_i \partial_i E_k) \quad \underline{\text{nice}}.$$
$$= e_k (\partial_k (\nabla \cdot E) - \nabla^2 E_k)$$
$$= \nabla(\nabla \cdot E) - \nabla^2 E$$

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3. Show that $\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$

Given $\nabla \cdot E = 0$, $\nabla \times E = -\partial_t B$ & $\nabla \times B = \mu_0 \epsilon_0 \partial_t E$

$$\nabla \times (\nabla \times E) = \nabla(\overset{0}{\cancel{\nabla \cdot E}}) - \nabla^2 E = -\nabla^2 E$$
$$\nabla \times (\nabla \times E) = \nabla \times (-\partial_t B) = -\partial_t (\nabla \times B) = -\partial_t (\mu_0 \epsilon_0 \partial_t E) = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$-\nabla^2 E = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\therefore \quad \nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \quad \checkmark$$

4. Given $\bar{x} = a(x - vt)$, $x = a(\bar{x} + v\bar{t})$

$\psi(\bar{t}) = a(\phi(t) - vt)$, $\phi(t) = a(\psi(\bar{t}) + v\bar{t})$

$\frac{d\phi}{dt} = \frac{d\psi}{d\bar{t}} = 1$

Derive $\bar{x} = \gamma(x - \beta t)$ & $\bar{t} = \gamma(t - \beta x)$:

$$1 = \frac{d}{dt}(\phi(t)) = \frac{d}{dt}\left[a(\psi(\bar{t}) + v\bar{t})\right] = a\left(\frac{d\psi}{d\bar{t}}\frac{d\bar{t}}{dt} + v\frac{d\bar{t}}{dt}\right) = a(1+v)\frac{d\bar{t}}{dt}$$

Also,

$$1 = \frac{d}{d\bar{t}}\psi(\bar{t}) = \frac{d}{d\bar{t}}\left[a(\phi(t) - vt)\right] = a\left(\frac{d\phi}{dt}\frac{dt}{d\bar{t}} - v\frac{dt}{d\bar{t}}\right) = a(1-v)\frac{dt}{d\bar{t}}^2$$

Multiplying the results together, we have

$$1 = a^2(1+v)(1-v)\underbrace{\frac{d\bar{t}}{dt}\frac{dt}{d\bar{t}}}_{=1}$$

$$1 = a^2(1-v^2)$$

$$\therefore a^2 = \frac{1}{1-v^2} \Rightarrow a = \gamma \equiv \frac{1}{\sqrt{1-v^2}} \qquad \text{let } v = \beta$$

then $a = \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$

By substitution,

$\bar{x} = a(x - vt)$ becomes $\bar{x} = \gamma(x - \beta t)$ ✓

$x = a(\bar{x} + v\bar{t})$ becomes $x = \gamma(\bar{x} + \beta\bar{t})$

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4. (cont'd) Next, need to show $\bar{t} = \gamma(t - \beta x)$

Take $x = \gamma(\bar{x} + \beta\bar{t})$ & solve for $\bar{t}$ $\Rightarrow \bar{t} = \frac{x}{\gamma\beta} - \frac{\bar{x}}{\beta}$

Plug in for $\bar{x}$: $\bar{t} = \frac{x}{\gamma\beta} - \frac{\gamma(x-\beta t)}{\beta} = \frac{x}{\gamma\beta} - \frac{\gamma x}{\beta} + \gamma t = \gamma t - (\frac{\gamma}{\beta} - \frac{1}{\gamma\beta})x$

Need to show $(\frac{\gamma}{\beta} - \frac{1}{\gamma\beta}) = \gamma\beta$

$$\frac{\gamma}{\beta} - \frac{1}{\gamma\beta} = \frac{\gamma^2}{\gamma\beta} - \frac{1}{\gamma\beta} = \frac{\gamma^2 - 1}{\gamma\beta}$$

Since $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, then $\gamma^2 = \frac{1}{1-\beta^2} \rightarrow \gamma^2(1-\beta^2) = 1 \rightarrow \gamma^2 - \gamma^2\beta^2 = 1$

Substitute value for 1 into $\frac{\gamma^2-1}{\gamma\beta}$: $\frac{\gamma^2 - (\gamma^2 - \gamma^2\beta^2)}{\gamma\beta} = \frac{\gamma^2 - \gamma^2 + \gamma^2\beta^2}{\gamma\beta} = \frac{\gamma^2\beta^2}{\gamma\beta} = \gamma\beta$

By substitution, plug in $\gamma\beta$ for $(\frac{\gamma}{\beta} - \frac{1}{\gamma\beta})$, so the equation for $\bar{t}$ becomes:

$$\bar{t} = \gamma t - (\frac{\gamma}{\beta} - \frac{1}{\gamma\beta})x = \gamma t - \gamma\beta x = \gamma(t - \beta x)$$

$$\therefore \bar{x} = \gamma(x - \beta t)$$
$$\bar{t} = \gamma(t - \beta x)$$ ✓

5. Given $\mathcal{Y} = A\mathcal{X} + tv + w$, show that if acceleration of $\gamma$ w.r.t $\mathcal{X}$ is zero then the acceleration of $\gamma$ w.r.t. $\mathcal{Y}$ is zero.

$$\mathcal{Y} = A\mathcal{X} + tv + w$$

$$\begin{aligned} a_y = \frac{d^2}{dt^2}(y) &= \frac{d}{dt}\left[\frac{d}{dt}\left(A\mathcal{X}(\gamma(t)) + tv + w\right)\right] \\ &= \frac{d}{dt}\left[A\frac{d}{dt}(\mathcal{X}(\gamma(t)) + v\right] \\ &= A\frac{d^2}{dt^2}(\mathcal{X}(\gamma(t))) \\ &= Aa_x \end{aligned}$$

If $a_x = 0$, then $a_y = Aa_x = A \cdot 0 = 0$.
Thus $a_x = 0 \Rightarrow a_y = 0$. ✓

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6. Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal transformation & $A \in \mathbb{R}^{3\times 3}$ is its matrix; $L(x) = Ax$. Show that $A^tA = I$.

Let $x, y \in \mathbb{R}^3$, then by definition of an orthogonal transformation

$$L(x) \cdot L(y) = x \cdot y$$

By defn, $x \cdot y = x^t y$, so

$$\begin{aligned} L(x) \cdot L(y) = x \cdot y &\Leftrightarrow (Ax)^t(Ay) = x^t y \\ &\Leftrightarrow x^tA^tAy = x^ty \\ &\Leftrightarrow x^t(A^tA)y = x^tIy \\ &\Leftrightarrow x^t(A^tA - I)y = 0 \checkmark \end{aligned}$$

$x^t(A^tA - I)y = 0 \ \forall\, x, y \in \mathbb{R}^3$ iff $A^tA - I = 0$ or $A^tA = I$.

7. Let $L: \mathbb{R}^4 \to \mathbb{R}^4$ be a Lorentz Transformation & $B \in \mathbb{R}^{4\times4}$ be its matrix; $L(x) = Bx$. Show that $B^T \eta B = \eta$.

- We know that $\langle x, y \rangle = x^T \eta y \quad \forall\ x, y \in \mathbb{R}^4$

- Now consider

$$\begin{aligned} \langle L(x), L(y) \rangle &= \langle Bx, By \rangle \\ &= (Bx)^t \eta By \\ &= x^t B^t \eta B y \\ &= x^t \eta y \end{aligned}$$

$\therefore\ \langle L(x), L(y) \rangle = \langle x, y \rangle \Leftrightarrow x^t B^t \eta B y = x^t \eta y$

which means that $B^t \eta B = \eta$.

✓