Math 231 Homework Project I: Vectors:

Follow instructions. Be careful to answer all the questions raised in each part. Please turn in neat work with problems clearly labeled and your name on each page. Thanks. Some of these problems are hard. You are not alone. If you get stuck, phone a friend. Or email me, or stop by office hours etc… Start early though, otherwise we may not be able to resolve your questions in time. Also, come to class. I may inadvertently work part of the homework. I plan for there to be 30 total problems. This assignment has 11 problems, this means it is worth 11pts of your final grade.

PROBLEM 1: Suppose that $\vec{A}, \vec{B}$ are non-zero two-dimensional vectors which are orthogonal. Show that if $\vec{C}$ is any two-dimensional vector then it can be written as a linear combination of the given vectors; that is show there exist $s, t \in \mathbb{R}$ such that $\vec{C} = s\vec{A} + t\vec{B}$. *Hint: if $\vec{u}$ is a unit vector then $\vec{u} \cdot \vec{C}$ gives the component of $\vec{C}$ in the $\vec{u}$ direction.*

PROBLEM 2: Suppose that $\vec{A}, \vec{B}$ are non-zero two-dimensional vectors which are not parallel. Show that if $\vec{C}$ is any two-dimensional vector then it can be written as a linear combination of the given vectors; that is show there exist $s, t \in \mathbb{R}$ such that $\vec{C} = s\vec{A} + t\vec{B}$.

PROBLEM 3: Inequalities of interest.

a. (Cauchy-Schwarz Inequality) Argue that $\vec{A} \cdot \vec{B} = AB\cos(\theta)$ implies $|\vec{A} \cdot \vec{B}| \leq |\vec{A}||\vec{B}|$.

b. (Triangle Inequality) Show that $|\vec{A} + \vec{B}| \leq |\vec{A}| + |\vec{B}|$. Hint: to accomplish this consider that $|\vec{A} + \vec{B}|^2 = (\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B})$ and use part a.

PROBLEM 4: Prove the following assertions:

a. $(\vec{v} \times \vec{w}) \cdot \vec{w} = 0$.

b. $(\vec{v} - \vec{w}) \times (\vec{v} + \vec{w}) = 2(\vec{v} \times \vec{w})$.

c. $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$.

d. $\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0$.

I would characterize parts a. and b. as fairly easy. Parts c. and d. are less easy. Part c. is not too hard to prove if you use the repeated index notation. Part d. follows from part c. and the commutativity of the dot-product. Part c. can also be done brute force without relying on the repeated index machinery, it is just tedious.

PROBLEM 5: Cancellation Properties: for ordinary real (or complex) numbers if $a \neq 0$ then $ab = ac$ implies that $b = c$. This problem investigates if that still is the case for the vectors' dot and cross products. Suppose that $\vec{A} \neq \vec{0}$ for the following questions.

a. If $\vec{A}\cdot\vec{B}=\vec{A}\cdot\vec{C}$ does it follow that $\vec{B}=\vec{C}$? If not give an argument why not. If true explain why it is true in general.

b. If $\vec{A}\times\vec{B}=\vec{A}\times\vec{C}$ does it follow that $\vec{B}=\vec{C}$? If not give an argument why not. If true explain why it is true in general.

c. If $\vec{A}\cdot\vec{B}=\vec{A}\cdot\vec{C}$ and $\vec{A}\times\vec{B}=\vec{A}\times\vec{C}$ does it follow that $\vec{B}=\vec{C}$? If not give an argument why not. If true explain why it is true in general.

PROBLEM 6: Find the equation of the plane that contains the points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$.

PROBLEM 7: Use Mathematica (or another program you prefer) to graph the following equations. Name each surface.

a. $x^2+4y^2+9z^2=0$,

b. $x=2y^2+3z^2$,

c. $x^2-y^2+z^2-2x+2y+4z+2=0$.

PROBLEM 8: Use Mathematica (or another program you prefer) to graph $z=f(x,y)$ for the following functions:

a. $f(x,y)=\sqrt{x^2+y^2}$,

b. $f(x,y)=e^{\sqrt{x^2+y^2}}$,

c. $f(x,y)=\sin(\sqrt{x^2+y^2})$,

d. $f(x,y)=\frac{1}{\sqrt{x^2+y^2}}$.

On the basis of those graphs, describe how the two-dimensional graph $y=g(x)$ and the three-dimensional graph $z=f(x,y)=g(\sqrt{x^2+y^2})$ are related.

PROBLEM 9: Calculus properties of vector-valued functions of a real variable. Prove each of the following assertions. Again the repeated index notation may be used if you find it helpful. Remember that to prove a vector equality we must prove equality for each component. We assume that f is a differentiable function of a real variable and $\vec{A},\vec{B},\vec{U},\vec{r}$ are vector-valued functions of a real variable such that all the derivatives listed below exist.

a. $\frac{d}{dt}(f\vec{U})=\frac{df}{dt}\vec{U}+f\frac{d\vec{U}}{dt}$

b. $\frac{d}{dt}(\vec{A}\cdot\vec{B})=\frac{d\vec{A}}{dt}\cdot\vec{B}+\vec{A}\cdot\frac{d\vec{B}}{dt}$

c. $\frac{d}{dt}(\vec{A}\times\vec{B})=\frac{d\vec{A}}{dt}\times\vec{B}+\vec{A}\times\frac{d\vec{B}}{dt}$

d. $\frac{d}{dt}[\vec{r}(t) \times \vec{r}'(t)] = \vec{r}(t) \times \vec{r}''(t)$ (use part c. plus a well-known fact about the cross product)

PROBLEM 10: Newton's 2nd Law is an equation that uses the calculus of vector-valued functions of a real variable. We are told that $\vec{F} = m\vec{a}$ for a particle with mass m and position $\vec{r} =< x, y, z >$ under the influence of a force $\vec{F}$. Recall that acceleration was defined to be the second derivative of position; $\vec{a}(t) = \vec{r}''(t)$. Find the position at time t given that the particle is subject to $\vec{F} =< kx, 0, -mg >$. State the answer in terms of the initial conditions $\vec{r}(0) =< x_o, y_o, z_o >$ and $\vec{v}(0) = \vec{r}'(0) =< v_{ox}, v_{oy}, v_{oz} >$.

PROBLEM 11: Kepler's Laws state that the planets orbit in elliptical paths. In other words, the motion of the planets lies in an orbital plane. Kepler made this assertion on the basis of experimental evidence gathered by Tycho Brahe. Later Isaac Newton postulated that gravity follows his universal law of gravitation. If we place the sun with mass M_s at the origin then the force on a planet of mass M_p at position $\vec{r}$ is given by the equation (G is the gravitation constant)

$$\vec{F}_{gravity} = \frac{-GM_sM_p\vec{r}}{r^3}$$

where $r = |\vec{r}|$ and the minus reflects the fact that gravity is an attractive force. For a particular planet we expect Newton's 2nd law should hold; $\vec{F}_{gravity} = M_p\vec{r}''$. Show that Newton's laws imply that the motion of the planet lies in some plane. A convenient characterization of a curve residing in a plane is that $\vec{r} \times \vec{r}' = \vec{c}$ for some constant vector $\vec{c}$.

(I would like to ask you to derive Kepler's Laws in their entirety but there is not quite enough time. It is not too hard, see my notes for a complete derivation, I doubt we will cover it in lecture this semester)