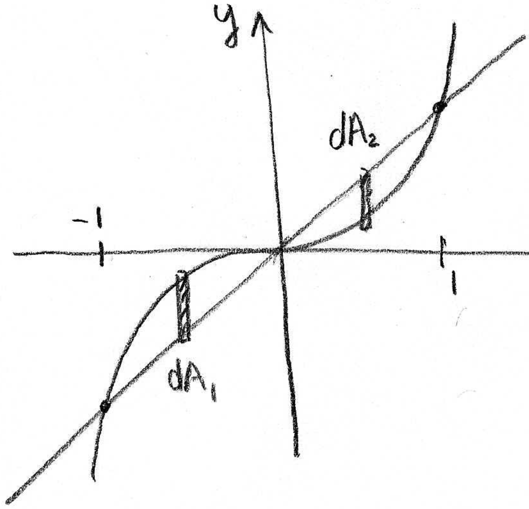
1.) Find area bounded by $y = x$ and $y = x^3$.

Note $x = x^3 \Rightarrow x^3 - x = x(x^2-1) = x(x+1)(x-1) = 0$

we have intersections at $x = 0$ & $x = \pm 1$.



$dA_1 = (x^3 - x)\,dx$ for $-1 \le x \le 0$.

$dA_2 = (x - x^3)\,dx$ for $0 \le x \le 1$.

$$A = \int_{-1}^{0} (x^3 - x)\,dx + \int_0^1 (x - x^3)\,dx$$

$$= \left(\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2\right)\Big|_{-1}^{0} + \left(\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right)\Big|_0^1$$

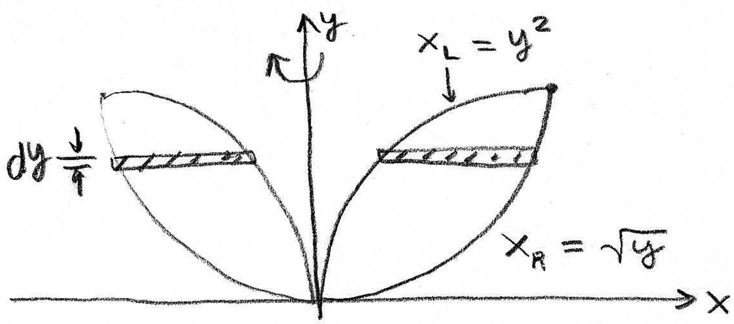
$$= \frac{-1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4}$$

$$= \boxed{\frac{1}{2}}$$

2.) Consider region bounded by $y = x^2$ and $y = \sqrt{x}$. Find volume of solid formed by rotating this region around various axis. Note $x^2 = \sqrt{x} \Rightarrow x^4 = x \Rightarrow x(x^3 - 1) = 0$

$\Rightarrow \underline{x = 0 \text{ \& } x = 1}$ intersection points.

(i) <u>rotate around $x = 0$</u>



$r_{in} = x_L = y^2$

$r_{out} = x_R = \sqrt{y}$

$$dV = \pi(r_{out}^2 - r_{in}^2)\,dy$$

$$= \pi(y - y^4)\,dy \quad \text{for } 0 \le y \le 1.$$

$$V = \int_0^1 \pi(y - y^4)\,dy = \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \boxed{\frac{3\pi}{10}}$$

MA 241-C08: Calculus II

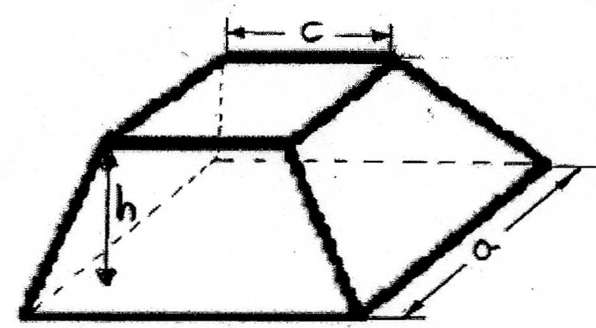
PRACTICE FOR TEST IV, MATH 131, 2008

Directions: You **must** show **ALL** your work to receive credit.

1. (25 pts) Find the area of the region bounded by $y = x$ and $y = x^3$. For full credit your solution should include a graph and diagrams of the appropriate infinitesimal rectangle(s).

2. (25pts) Let us give the region bounded by $y = x^2$ and $y = \sqrt{x}$ the name *Dwight* . Find the volume of the solid obtained from rotating *Dwight* around the following:
   (i.) $x = 0$
   (ii.) $x = a, \quad a < 0$
   (iii.) $x = a, \quad a \geq 1$
   (iv.) $y = 0$
   (v.) $y = b, \quad b < 0$
   (vi.) $y = b, \quad b \geq 1$
   Do the above one at a time. On the test I'll pick one of the cases above. Again for full credit your solution should include a graph of the region as well as some picture of the typical approximating washer with r_{in} and r_{out} as they relate to the region in question (could be y_b or y_T or x_L or x_R as we have seen in examples). In other words, present your solution roughly as I have in class, try to indicate where the final integral came from. The set-up is worth most of the points on a problem like this.

4. **(15pts) Find volume of a square pyramid which has its top cut off at height h. Suppose that the each side of the base of pyramid has length a. The top of the shape is coplanar to the base and the top forms a square with sides of length c. Your answer will be in terms of the constants a, c and h.**



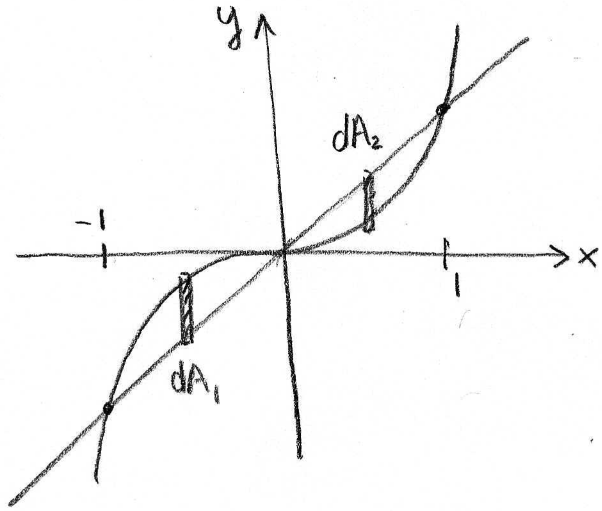
3.) Find the position X and velocity V as functions of time given $a(t) = t$, $X(0) = 4$ and $V(0) = 7$.

5.) $\int_0^1 \cos(x)\, e^{\sin(x)}\, dx$, $\int \cos^4(x) \sin(x)\, dx$, $\int \frac{dx}{\sqrt{4 - x^2}}$, $\int e^x \cos(e^x)\, dx$

$\int_a^b z e^{z^2}\, dz$, $\int \tan^2(x)\, dx$, $\int x\sqrt{x-3}\, dx$, $\int \cot(\theta)\, d\theta$

FALL 2008, PRACTICE PROBLEMS FOR TEST IV

1.) Find area bounded by $y = x$ and $y = x^3$.
Note $x = x^3 \Rightarrow x^3 - x = x(x^2-1) = x(x+1)(x-1) = 0$
we have intersections at $x = 0$ & $x = \pm 1$.

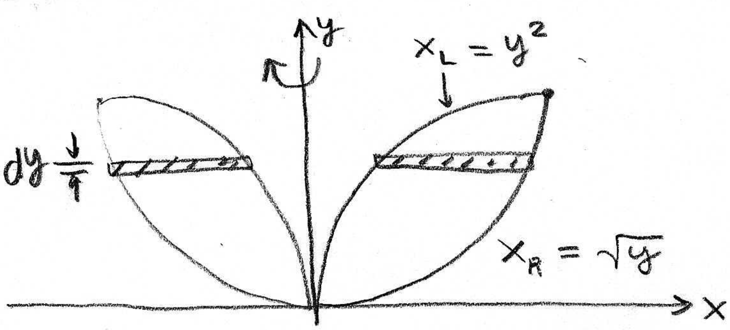


$dA_1 = (x^3 - x)\,dx$ for $-1 \le x \le 0$.
$dA_2 = (x - x^3)\,dx$ for $0 \le x \le 1$.

$$A = \int_{-1}^{0} (x^3 - x)\,dx + \int_0^1 (x - x^3)\,dx$$
$$= \left(\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2\right)\Big|_{-1}^{0} + \left(\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right)\Big|_0^1$$
$$= \frac{-1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4}$$
$$= \boxed{\frac{1}{2}}$$

2.) Consider region bounded by $y = x^2$ and $y = \sqrt{x}$. Find volume of solid formed by rotating this region around various axis. Note $x^2 = \sqrt{x} \Rightarrow x^4 = x \Rightarrow x(x^3-1) = 0$
$\Rightarrow$ $x = 0$ & $x = 1$ intersection points.

(i) rotate around $x = 0$



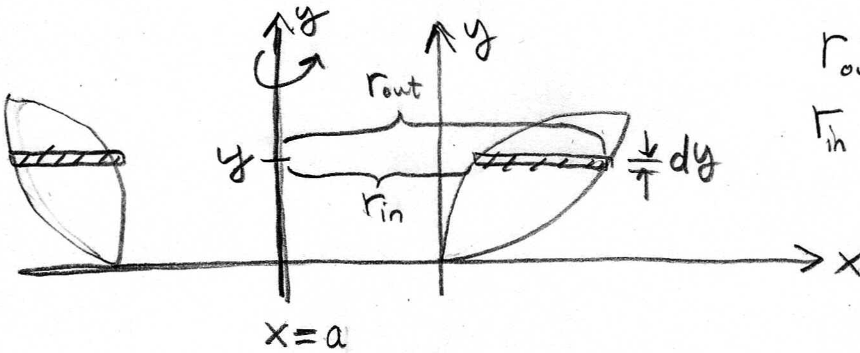
$r_{in} = x_L = y^2$
$r_{out} = x_R = \sqrt{y}$

$$dV = \pi(r_{out}^2 - r_{in}^2)\,dy$$
$$= \pi(y - y^4)\,dy \quad \text{for } 0 \le y \le 1.$$

$$V = \int_0^1 \pi(y - y^4)\,dy = \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \boxed{\frac{3\pi}{10}}$$

2.) <u>Continued</u>:

ii) Let $a < 0$

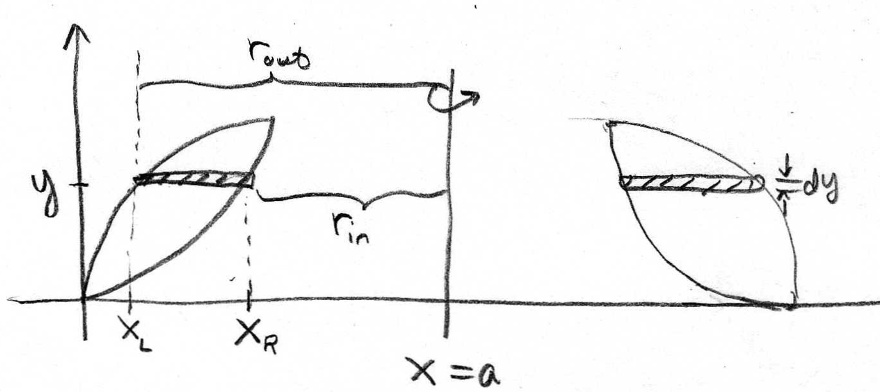


$$r_{out} = x_R - a = \sqrt{y} - a$$
$$r_{in} = x_L - a = y^2 - a$$

$$dV = \pi\left(r_{out}^2 - r_{in}^2\right)dy = \pi\left((\sqrt{y}-a)^2 - (y^2-a)^2\right)dy$$
$$V = \int_0^1 \pi\left([\sqrt{y}-a]^2 - [y^2-a]^2\right)dy = \boxed{\pi\left(\frac{3}{10} - \frac{2a}{3}\right)}$$

<u>Remark</u>: I'm skipping steps for the integrals in this problem, they're not difficult, if you used specific numerical values for a then you can check your answer with my general formulas. For example rotation around $x=-1$ would yield $V = \pi\left(\frac{3}{10} + \frac{2}{3}\right) = \frac{29\pi}{30}$.

iii) Let $a \geq 1$

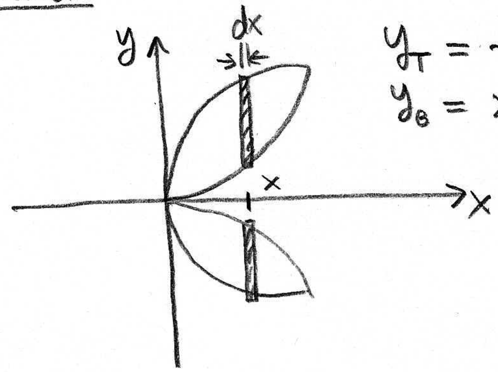


$$r_{in} = a - \sqrt{y}$$
$$r_{out} = a - y^2$$

$$dV = \pi\left(r_{out}^2 - r_{in}^2\right)dy = \pi\left([a-y^2]^2 - [a-\sqrt{y}]^2\right)dy$$
$$V = \int_0^1 \pi\left([a-y^2]^2 - [a-\sqrt{y}]^2\right)dy = \boxed{\pi\left(\frac{2a}{3} - \frac{3}{10}\right)}$$

2.) Continued

iv.)



$y_T = \sqrt{x}$
$y_B = x^2$

$r_{out} = y_T = \sqrt{x}$
$r_{in} = y_B = x^2$

$$dV = \pi(r_{out}^2 - r_{in}^2)\,dx = \pi(x - x^4)\,dx$$

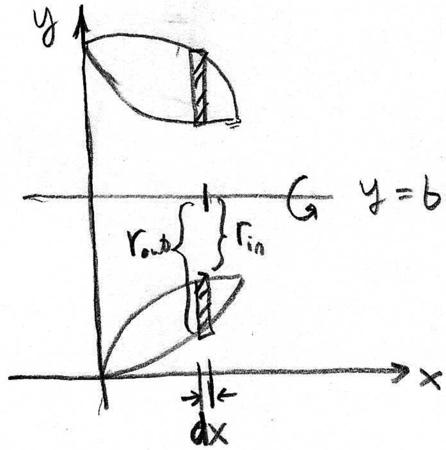
$$V = \int_0^1 \pi(x - x^4)\,dx$$

$$= \pi\left(\tfrac{1}{2} - \tfrac{1}{5}\right)$$

$$= \boxed{\frac{3\pi}{10}}$$

(oops)

vi.) $b > 1$

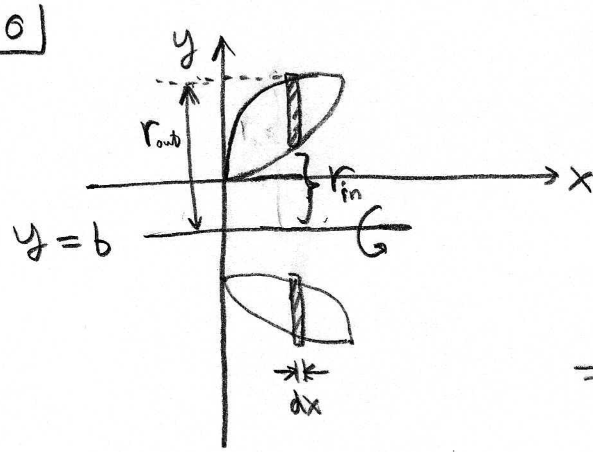


$r_{in} = b - y_T = b - \sqrt{x}$
$r_{out} = b - y_B = b - x^2$

$$dV = \pi(r_{out}^2 - r_{in}^2)\,dx$$

$$V = \int_0^1 \pi\left([b - x^2]^2 - [b - \sqrt{x}]^2\right)dx = \boxed{\pi\left(\frac{2b}{3} - \frac{3}{10}\right)}$$

v.) $b < 0$



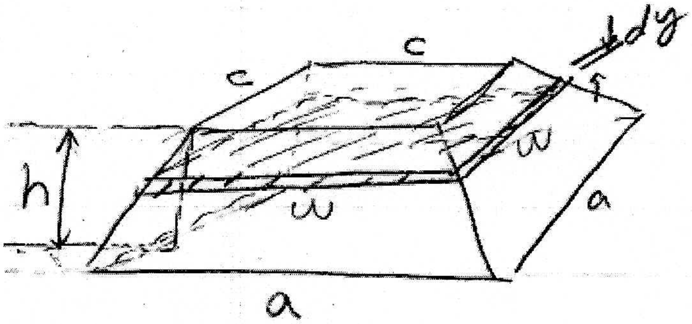
$r_{in} = y_B - b = x^2 - b$
$r_{out} = y_T - b = \sqrt{x} - b$

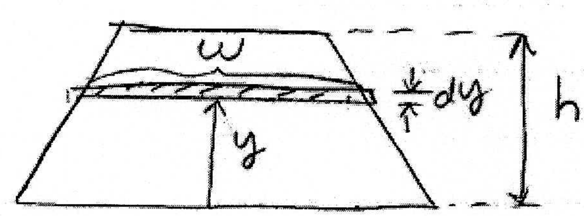
$$dV = \pi(r_{out}^2 - r_{in}^2)\,dx$$

$$\Rightarrow \boxed{V = \pi\left(\frac{3}{10} - \frac{2b}{3}\right)}$$

Remark: this is a quirky example, usually we would not find so much symmetry between the cases. The area "Dwight" is quite special.

PROBLEM FOUR





$$dV = w^2 dy$$

$$dV = \left[\left(\frac{c-a}{h}\right)y + a\right]^2 dy$$

$w = my + b$

$a = m(0) + b \therefore \underline{b = a}$

$c = mh + a \therefore m = \frac{c-a}{h}$

$$V = \int_0^h \left[\left(\frac{c-a}{h}\right)y + a\right]^2 dy$$

$$= \int_0^h \left[\left(\frac{c-a}{h}\right)^2 y^2 + 2a\left(\frac{c-a}{h}\right) + a^2\right] dy$$

$$= \frac{1}{3}\left(\frac{c-a}{h}\right)^2 h^3 + a\left(\frac{c-a}{h}\right)h^2 + a^2 h$$

$$= \boxed{\frac{1}{3}(c-a)^2 h + a(c-a)h + a^2 h}$$

3.) Let $a(t) = t$ and suppose $x(0) = 4$ & $V(0) = 7$.

$a = \frac{dV}{dt} = t \Rightarrow V = \frac{1}{2}t^2 + C_1$ (integrate indefinitely)

Note, $V(0) = 7 = \frac{1}{2}(0)^2 + C_1 \Rightarrow \underline{C_1 = 7}$

$\therefore \boxed{V(t) = \frac{1}{2}t^2 + 7}$

Next, $V = \frac{dx}{dt} = \frac{1}{2}t^2 + 7$.

Integrate $\Rightarrow x = \frac{1}{6}t^3 + 7t + C_2$

Note, $x(0) = \underline{4 = C_2}$ $\therefore \boxed{x(t) = \frac{1}{6}t^3 + 7t + 4}$

(5)

5.)
$$\int_0^1 e^{\sin(x)}\cos(x)\,dx = \int_0^{\sin(1)} e^u\,du$$

$u = \sin(x)$ $\quad u(0) = 0$
$du = \cos(x)dx$ $\quad u(1) = \sin(1)$

$$= e^u\Big|_0^{\sin(1)}$$
$$= \boxed{e^{\sin(1)} - 1} \qquad (e^0 = 1)$$

$$\int \cos^4(x)\sin(x)\,dx = \int u^4(-du)$$

$\left(\begin{array}{l} u = \cos(x) \\ du = -\sin(x)dx \end{array}\right)$

$$= \frac{-1}{5}u^5 + C$$
$$= \boxed{\frac{-1}{5}\cos^5(x) + C}$$

$$\int \frac{dx}{\sqrt{4-x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{1-x^2/4}}$$
$$= \frac{1}{2}\int \frac{2\,du}{\sqrt{1-u^2}}$$

$u = x/2$
$du = dx/2$

$$= \sin^{-1}(u) + C$$
$$= \boxed{\sin^{-1}(x/2) + C}$$

$$\int e^x\cos(e^x)\,dx = \int \cos(u)\,du$$

$u = e^x$
$du = e^x dx$

$$= \sin(u) + C$$
$$= \boxed{\sin(e^x) + C}$$

$$\int_a^b z\,e^{z^2}dz = \int_{a^2}^{b^2} \frac{1}{2}e^u\,du$$

$u = z^2$ $\quad du = 2z\,dz$
$u(a) = a^2$
$u(b) = b^2$ change bounds!

$$= \boxed{\frac{1}{2}\left(e^{b^2} - e^{a^2}\right)}$$

5. Continued

$$\int \tan^2(x)\,dx = \int (\sec^2(x) - 1)\,dx$$

$$= \int \sec^2(x)\,dx - \int dx$$

$$= \boxed{\tan(x) - x + C}$$

$$\int x\sqrt{x-3}\,dx = \int (u+3)\sqrt{u}\,du \quad \leftarrow \boxed{\begin{array}{l} u = x-3 \\ du = dx \\ x = u+3 \end{array}}$$

$$= \int \left(u^{3/2} + 3u^{1/2}\right)du$$

$$= \frac{2}{5}u^{5/2} + \frac{2}{3}3u^{3/2} + C$$

$$= \boxed{\frac{2}{5}(x-3)^{5/2} + 2(x-3)^{3/2} + C}$$

$$\int \cot(\theta)\,d\theta = \int \frac{\cos\theta}{\sin\theta}\,d\theta$$

$$= \int \frac{du}{u} \quad \leftarrow \boxed{\begin{array}{l} u = \sin\theta \\ du = \cos\theta\,d\theta \end{array}}$$

$$= \ln|u| + C$$

$$= \boxed{\ln|\sin\theta| + C}$$