Math 131, Test 2, September 30, 2008 Name: ______________.

No graphing calculators or electronic communication of any kind. If you need extra paper please ask. Credit will be awarded for correct content and clarity of presentation. If proper notation is not present in your solution you will likely lose at least 3pts for each offense. This test has 105 points, 5 are bonus points. Make sure to at least attempt each part.

1. [10pts a piece worth either 30pts or 50pts, your choice.] Complete the following differentiations (**do not simplify**): If you would rather complete all 5 parts here you can skip problem 2. On the other hand you need only do 3 of these if you choose to do problem 2. Please mark out the parts you do NOT want graded. (no bonus to be had here). Find $\frac{dy}{dx}$ for each of the following problems you choose:

a.) $y = x^3 + e^x + 3^x + \sin^{-1}(x) + 7\pi$

$$\frac{dy}{dx} = \boxed{3x^2 + e^x + \ln(3)\, 3^x + \frac{1}{\sqrt{1-x^2}}}$$

b.) Suppose A, B are constants, $y = A(x^2 + Bx)^3$

$$\frac{dy}{dx} = A\frac{d}{dx}(x^2 + Bx)^3$$

$$= 3A(x^2 + Bx)^2 \frac{d}{dx}(x^2 + Bx)$$

$$= \boxed{3A(x^2 + Bx)^2(2x + B)}$$

c.) $y = \sin(x)\cos(\sqrt{x})$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin(x))\cos(\sqrt{x}) + \sin(x)\frac{d}{dx}(\cos(\sqrt{x}))$$

$$= \cos(x)\cos(\sqrt{x}) - \sin(x)\sin(\sqrt{x})\frac{d}{dx}(\sqrt{x})$$

$$= \boxed{\cos(x)\cos(\sqrt{x}) - \frac{\sin(x)\sin(\sqrt{x})}{2\sqrt{x}}}$$

d.) $y = x^{\sin(x)}$

$$\ln(y) = \ln\left(x^{\sin(x)}\right) = \sin(x)\ln(x)$$

$$\frac{1}{y}\frac{dy}{dx} = \cos(x)\ln(x) - \sin(x)\left(\frac{1}{x}\right)$$

$$\boxed{\frac{dy}{dx} = x^{\sin(x)}\left(\ln(x)\cos(x) - \frac{1}{x}\sin(x)\right)}$$

e.) $y = \dfrac{ae^x + b}{ce^x + d} + \tan^{-1}(\cos(x))$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{ae^x + b}{ce^x + d}\right) + \frac{d}{dx}\left(\tan^{-1}(\cos(x))\right)$$

$$= \frac{ae^x(ce^x + d) - (ae^x + b)(ce^x)}{(ce^x + d)^2} + \frac{1}{1 + \cos^2(x)}\frac{d}{dx}\left(\cos(x)\right)$$

$$= \boxed{\frac{(ad - bc)e^x}{(ce^x + d)^2} - \frac{\sin(x)}{1 + \cos^2(x)}}$$

2. [20pts or 0pts, your choice] Prove the following: If functions f(x) and g(x) are differentiable and positive everywhere then

$$\frac{d}{dx}\left(fg\right) = \frac{df}{dx}g + f\frac{dg}{dx}.$$

Let $y = fg$ then

$$\ln(y) = \ln(fg) = \ln(f) + \ln(g)$$

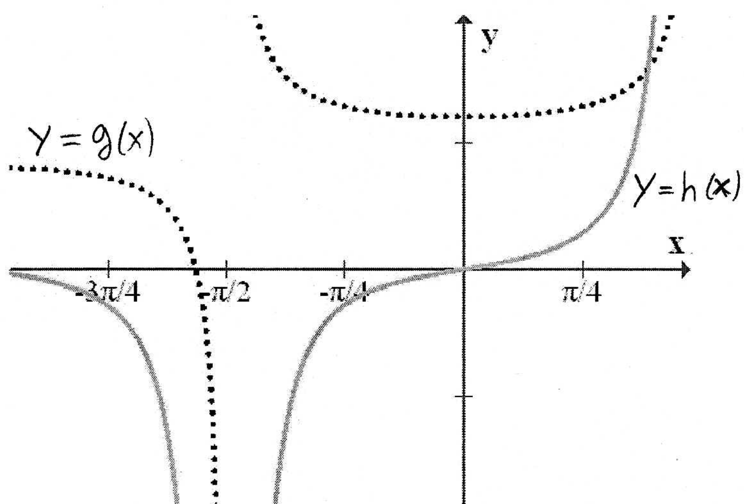
Differentiate implicitly with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{f}\frac{df}{dx} + \frac{1}{g}\frac{dg}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{fg}{f}\frac{df}{dx} + \frac{fg}{g}\frac{dg}{dx}$$

$$\therefore \frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx} = \underline{\frac{df}{dx}g + f\frac{dg}{dx}}.$$

3. [6pts] Explain which of the graphs is $y = f(x)$ and which is $y = \frac{df}{dx}$. Give analytical reasons for your choice. Let's define that dotted graph is y = g(x) while the un-dotted graph is y = h(x). So the question is whether f = g or f = h ?



$g(x) = \sec(x) + 5$

$h(x) = \sec(x)\tan(x)$

So clearly we have

$f(x) = g(x)$

$\frac{df}{dx} = h(x)$

Now you weren't supposed to see that this was the graph of secant, you should have observed:

1.) $h(x) > 0$ wherever $g(x)$ is increasing

2.) $h(x) < 0$ wherever $g(x)$ is decreasing

3.) $h(x) = 0$ at horizontal tangents of $y = g(x)$.

Thus, $h(x) = \frac{df}{dx}$ and $g(x) = f(x)$.

4.[14pts] Find the equation of the tangent line at (1,1) to the ellipse,

$$x^2 + 2y^2 = 3.$$

$$2x + 4y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{2y}$$

Now at $(1,1)$ we find $\frac{dy}{dx}(1,1) = \frac{-1}{2(1)} = \frac{-1}{2}$.

This is the slope of the tangent line through $(1,1)$

thus, using point-slope:

$$\boxed{y = 1 - \frac{1}{2}(x-1)}.$$

5. [10pts] Show that $\frac{d}{dx}(\ \tan^{-1}(x)\ ) = \frac{1}{1+x^2}$.

Let $y = \tan^{-1}(x)$ then $\tan(y) = x$ thus

$$\sec^2(y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)}$$

$$= \frac{1}{1+\tan^2(y)}$$

$$= \frac{1}{1+x^2}$$

$$\therefore \boxed{\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}}.$$

6.[15pts] Let $y = p(x)$ be the graph of a quadratic polynomial. If we are given that

$$p(1) = 2 \qquad \frac{dp}{dx}(1) = \pi \qquad \frac{d^2p}{dx^2}(1) = 2$$

then find the explicit formula for $p(x)$.

$$P(x) = Ax^2 + Bx + C$$

$$P'(x) = 2Ax + B$$

$$P''(x) = 2A$$

Now we need to find A, B, C using the given data,

$$P''(1) = 2 = 2A \longrightarrow \underline{A = 1}.$$

$$P'(1) = 2 + B = \pi \longrightarrow \underline{B = \pi - 2}.$$

$$P(1) = A + B + C = 2$$

$$1 + \pi - 2 + C = 2$$

$$\Rightarrow \underline{C = 3 - \pi}.$$

Thus,

$$\boxed{P(x) = x^2 + (\pi - 2)x + 3 - \pi}$$

7. [5pts] Show that $\frac{d}{dx}(\ \tan(x)\ ) = \sec^2(x)$.

$$\frac{d}{dx}\left(\tan(x)\right) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$$

$$= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$$

$$= \frac{1}{\cos^2(x)} \quad \therefore \quad \boxed{\frac{d}{dx}\left(\tan(x)\right) = \sec^2(x)}$$

8.[BONUS][5pts]

a.) [1pt] If x=5 what is x ?

X is 5.

b.) [4pts] Given that

$$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix}) \qquad \sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$$

find the values of A, B, C, D such that the following identity holds true

$$\sin(3x)\sin(4x) = A\cos(Bx) + C\cos(Dx).$$

$$\sin(3x)\sin(4x) = \frac{1}{2i}\left(e^{3ix} - e^{-3ix}\right)\frac{1}{2i}\left(e^{4ix} - e^{-4ix}\right)$$

$$= \frac{-1}{4}\left(e^{7ix} - e^{-ix} - e^{ix} + e^{-7ix}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\left[e^{ix} - e^{-ix}\right]\right) - \frac{1}{2}\left(\frac{1}{2}\left[e^{7ix} - e^{-7ix}\right]\right)$$

$$= \frac{1}{2}\cos(x) - \frac{1}{2}\cos(7x)$$

Thus, $\boxed{A = \frac{1}{2},\ B = 1,\ C = \frac{-1}{2},\ D = 7}$

OR, $\boxed{A = \frac{-1}{2},\ B = 7,\ C = \frac{1}{2},\ D = 1}$