§12.3 #7 Consider $S = \sum_{n=1}^{\infty} ne^{-n}$. Use the $\int$-test to prove convergence or divergence of S.

(1) $f(x) = xe^{-x}$ is positive and $\lim_{x\to\infty}(xe^{-x}) = \lim_{x\to\infty}\left(\frac{x}{e^x}\right) = 0$

By L'Hopital's Rule get $\lim_{x\to\infty}\left(\frac{1}{e^x}\right) = 0$

Note $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$

thus f is decreasing on $[1,\infty)$

since $f'(x) \leq 0$ for $x \geq 1$.

Thus the $\int$-test applies.

$$\int_1^{\infty} xe^{-x}\,dx = \lim_{t\to\infty}\int_1^t xe^{-x}\,dx$$
$$= \lim_{t\to\infty}\left[\left(-xe^{-x} - e^{-x}\right)\right]_1^t$$
$$= \lim_{t\to\infty}\left[-te^{-t} - \underbrace{e^{-t}}_{\to 0} + e^{-1} + e^{-1}\right]$$
$$= \frac{2}{e} - \lim_{t\to\infty}\left(\frac{t}{e^t}\right)$$
$$= \frac{2}{e} - \underbrace{\lim_{t\to\infty}\left[\frac{1}{e^t}\right]}_{\to 0}$$
$$= \frac{2}{e}$$
(converges $\therefore \sum_{n=1}^{\infty} ne^{-n}$ converges.)

Remark: a better way to do this is the ratio test,

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{(n+1)e^{-(n+1)}}{ne^{-n}}\right) = \lim_{n\to\infty}\left(\frac{n+1}{n}\left(\frac{1}{e}\right)\right) = \frac{1}{e}$$

Thus $L = 1/e < 1$ $\therefore$ the series converges by ratio test.

§12.3 #12 | Conv / Div ?

$$S = 1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$$ converges by ~~[illegible]~~ $p=1.5$ series test.

§12.3 #17 | Conv / div ?

$$\sum_{n=1}^{\infty} \frac{1}{n^2+4}$$

Notice that $\frac{1}{n^2+4}$ corresponds to $f(x) = \frac{1}{x^2+4}$ which is clearly positive, $f'(x) = \frac{-2x}{(x^2+4)^2} < 0$ for $x > 1$ and $\lim_{x\to\infty}\left(\frac{1}{x^2+4}\right) = 0$ $\therefore$ $\int$ - test applies

$$\int_1^{\infty} \frac{dx}{x^2+4} = \lim_{t\to\infty}\left(\int_1^t \frac{1}{4}\left[\frac{dx}{1+(x/2)^2}\right]\right)$$

$u = x/2$
$du = dx/2$
$2du = dx$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$$ $\therefore$ $\int$-converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+4}$ conv. by $\int$ - test.

Alternatively:

$\frac{1}{n^2+4} \leq \frac{1}{n^2}$ $\therefore$ $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ converges

By the D.C.T. to the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ($p=2$ series)

§12.3 #31 | Find values for p for which $\sum_{n=1}^{\infty} \frac{1}{n[\ln(n)]^p}$ converges.

[How? Use the $\int$ - test and study the integral $\int u^{-p}\,du$ for $u = \ln(x)$.
I leave this to the students.]

§12.3 #34 | I also leave to the reader.

§12.2 #67 Prove part (i.) of Th^{m} 8; prove $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

We're given $\sum_{n=1}^{\infty} a_n$ converges to L for some $L \in \mathbb{R}$.

Thus $S_n = \sum_{k=1}^{n} a_k$ is a convergent sequence and

$\lim_{n\to\infty}(S_n) = L$. Consider that

$$\sum_{k=1}^{n} ca_k = c\left(\sum_{k=1}^{n} a_k\right) = c\,S_n$$

Therefore, $\sum_{n=1}^{\infty} ca_n = \lim_{n\to\infty}(c\,S_n) = c\lim_{n\to\infty}(S_n) = cL = c\sum_{n=1}^{\infty} a_n$ //

§12.2 #68 Prove if $\sum a_n$ is divergent then $\sum ca_n$ is divergent provided that $c \neq 0$

Again observe that the n^{th} partial sum for $\sum ca_n$ is simply $c\,S_n$ where S_n is the n^{th} partial sum for $\sum a_n$. But $\lim_{n\to\infty} S_n$ diverges iff $\lim_{n\to\infty}(c\,S_n)$ diverges provided $c \neq 0$. The theorem follows.

§12.2 #70 The answer is no. Consider that $\sum_{n=1}^{\infty} \frac{1}{n}$ & $\sum_{n=1}^{\infty} \frac{-1}{n}$ both diverge

but $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n}\right) = \sum_{n=1}^{\infty} 0 = 0$ (converges.)

Thus the convergence of $\sum (a_n + b_n)$ cannot insure the convergence of $\sum a_n$ AND $\sum b_n$.

(I just gave a counter-example)