HOMEWORK 24 : CALCULUS II : §12.10 # 9, 11, 15, 25, 28, 29, 33, 36, 39, 41, 47, 50, 59
(STEWART 6th Ed.)

§12.10 #9 $f(x) = e^{5x}$ find Maclaurin series from Taylor's Expansion directly

Note that $f'(x) = 5e^{5x}$ then $f''(x) = 5(5e^{5x}) = 5^2e^{5x}$ then $f'''(x) = 5^3e^{5x} \Rightarrow f^{(n)}(x) = 5^n e^{5x}$ $\therefore$ $f^{(n)}(0) = 5^n e^0 = 5^n$.

$$e^{5x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \boxed{\sum_{n=0}^{\infty} \frac{5^n}{n!}x^n = 1 + 5x + \cdots}$$

Remark: if we already knew that $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$ then we could just substitute $u = 5x$ and find the same answer quicker. However, Stewart asked us to use Taylor's Thm directly.

Radius of Convergence? Examine ratio test,

$$L = \lim_{n\to\infty}\left|\frac{5^{n+1}(x)^{n+1}}{(n+1)!}\frac{n!}{5^n(x)^n}\right|$$
$$= \lim_{n\to\infty}\left(\frac{5}{n+1}\right)|x|$$
$$= 0 \quad \therefore \quad \boxed{R.O.C. = \infty \text{ since } I.O.C. = \mathbb{R}}$$

§12.10 #11 Find Maclaurin series for $f(x) = \sinh(x)$ and find the R.O.C.

$f(x) = \sinh(x) \Rightarrow f(0) = 0$
$f'(x) = \cosh(x) \Rightarrow f'(0) = 1$
$f''(x) = \sinh(x) \Rightarrow f''(0) = 0$
$f'''(x) = \cosh(x) \Rightarrow f'''(0) = 1$

Thus $f^{(2n+1)}(0) = 1$ whereas $f^{(2n)}(0) = 0$ for $n = 0, 1, 2, \dots$

$$\therefore \boxed{\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots}$$

The R.O.C. comes from examining ratio test,

$$L = \lim_{n\to\infty}\left|\frac{x^{2(n+1)+1}}{[2(n+1)+1]!}\frac{(2n+1)!}{x^{2n+1}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{x^{2n+3}}{x^{2n+1}}\frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!}\right|$$
$$= \lim_{n\to\infty}\left(|x^2|\left|\frac{1}{(2n+3)(2n+2)}\right|\right) = 0 \quad \therefore \quad \boxed{R.O.C. = \infty}$$

§12.10 #15 Find Taylor series centered at $a = 3$ for $f(x) = e^x$

Notice $f^{(n)}(x) = e^x$ since $\frac{d}{dx}(e^x) = e^x \Rightarrow f^{(n)}(x) = f^{(n+1)}(x)$ etc...

Thus $f^{(n)}(\cancel{0}3) = e^3$ thus,

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n = e^3\left(1 + (x-3) + \tfrac{1}{2}(x-3)^2 + \cdots\right)}$$

Remark: we could also obtain this in a sneaky way,

$$e^x = e^{x-3+3} = e^3 e^{x-3} = e^3 \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}$$

§12.10 #25 Use the binomial series to expand $\sqrt{1+x}$ as a power series. State the R.O.C.

$$\begin{aligned}
\sqrt{1+x} &= (1+x)^{1/2} \\
&= 1 + \tfrac{1}{2}x + \tfrac{1}{2}\left(\tfrac{1}{2}-1\right)x^2 + \tfrac{1}{3!}\,\tfrac{1}{2}\left(\tfrac{1}{2}-1\right)\left(\tfrac{1}{2}-2\right)x^3 + \cdots \\
&= 1 + \tfrac{1}{2}x - \tfrac{1}{4}x^2 + \tfrac{1}{3\cdot 2}\left(\tfrac{1}{2}\right)\left(\tfrac{-1}{2}\right)\left(\tfrac{-3}{2}\right)x^3 + \cdots \\
&= \boxed{1 + \tfrac{1}{2}x - \tfrac{1}{4}x^2 + \tfrac{1}{16}x^3 + \cdots = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n}
\end{aligned}$$

the R.O.C. = 1 by the "Binomial Series thm" (debateable notation)

(we'll discuss in lecture why R.O.C. = 1)

§12.10 #28 Use binomial series to expand $(1-x)^{2/3}$ as power series, state R.O.C.

Let $u = -x$ then

$$\begin{aligned}
(1-x)^{2/3} &= (1+u)^{2/3} \\
&= 1 + \tfrac{2}{3}u + \tfrac{1}{2}\left(\tfrac{2}{3}\right)\left(\tfrac{-1}{3}\right)u^2 + \tfrac{1}{3!}\left(\tfrac{2}{3}\right)\left(\tfrac{-1}{3}\right)\left(\tfrac{-4}{3}\right)u^3 + \cdots \\
&= 1 + \tfrac{2}{3}(-x) - \tfrac{1}{9}(-x)^2 + \tfrac{8}{3^4\cdot 2}(-x)^3 + \cdots \\
&= \boxed{1 - \tfrac{2}{3}x - \tfrac{1}{9}x^2 - \tfrac{4}{81}x^3 - \cdots, \quad \text{R.O.C.} = 1}
\end{aligned}$$

§12.10 #29 Use known Maclaurin series to expand $f(x) = \sin(\pi x)$

$$\sin(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\pi x)^{2n+1} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} x^{2n+1}}$$

Remark: Problems 25 & 28 don't allow for a nice clean answer like this

§12.10 #33 Expand $f(x) = x\cos\left(\frac{1}{2}x^2\right)$ via the known cosine expansion

$$
\begin{aligned}
f(x) &= x\cos\left(\tfrac{1}{2}x^2\right) \\
&= x\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\tfrac{1}{2}x^2\right)^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{2}\right)^{2n} x\cdot x^{4n} \\
&= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(2n)!}\, x^{4n+1} = x - \frac{1}{8}x^9 + \cdots}
\end{aligned}
$$

§12.10 #36 Use known expansions to obtain power series for $f(x) = \dfrac{x^2}{\sqrt{2+x}}$

Idea: this is almost $(1+u)^k$ with $k = -1/2$ but the x^2 rides along and the 2 is annoying. Let's make it go away,

$$
f(x) = \frac{x^2}{\sqrt{2+x}} = \frac{x^2}{\sqrt{2(1+x/2)}} = \frac{x^2}{\sqrt{2}}\underbrace{\left(1+\frac{x}{2}\right)^{-1/2}}_{\substack{\text{apply binomial series}\\ \text{to this part, } u = \frac{x}{2}}}
$$

$$
\begin{aligned}
\Rightarrow f(x) &= \frac{x^2}{\sqrt{2}}\left(1 - \tfrac{1}{2}u + \tfrac{1}{2!}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)u^2 + \tfrac{1}{3!}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)u^3 + \cdots\right) \\
&= \frac{x^2}{\sqrt{2}}\left(1 - \tfrac{1}{2}\left(\tfrac{x}{2}\right) + \tfrac{3}{8}\left(\tfrac{x}{2}\right)^2 - \tfrac{1}{3\cdot 2}\left(\tfrac{3\cdot 5}{8}\right)\left(\tfrac{x}{2}\right)^3 + \cdots\right) \\
&= \boxed{\frac{1}{\sqrt{2}}\left(x^2 - \frac{1}{4}x^3 + \frac{3}{32}x^4 - \frac{5}{128}x^5 + \cdots\right)}
\end{aligned}
$$

§12.10 #39 Find Maclaurin series for $f(x) = \cos(x^2)$

$$
\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x^2)^{2n} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{4n}}
$$

(If you look at $y = \cos(x^2)$ and compare to $y = 1 - \frac{1}{2}x^4$ and then $y = 1 - \frac{1}{2}x^4 + \frac{1}{24}x^8$ you should see that the graphs of $y = T_4(x)$ then $y = T_8(x)$ get closer & closer to $y = \cos(x^2)$.)

§12.10 #41 Find Maclaurin series $f(x) = xe^{-x}$

$$
f(x) = xe^{-x} = x\sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = x\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^{n+1}}
$$

(similar comments to #39 apply here.)

§12.10 # 47 Find power series sol$^{\underline{n}}$ of the integral below:

$$\int x\cos(x^3)\,dx$$

Notice $x\cos(x^3) = x\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x^3)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{6n+1}$. Thus,

$$\int x\cos(x^3)\,dx = \int\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{6n+1}\right)dx$$

$$= \underline{\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(2n)!}\,\frac{x^{6n+2}}{6n+2}\right] + C}.$$

$$= \underline{\left(\frac{1}{2}x^2 - \frac{1}{16}x^8 + \frac{1}{(24)(14)}x^{14} + \cdots\right) + C}.$$

§12.10 # 50 Find series sol$^{\underline{n}}$ for $\int \tan^{-1}(x^2)\,dx$

Note $f(x) = \tan^{-1}(x^2) \Rightarrow \frac{df}{dx} = \frac{2x}{1+x^2} = \sum_{n=0}^{\infty}2x(-x^2)^n = \sum_{n=0}^{\infty}2(-1)^n x^{2n+1}$

Hence $f(x) = C + \sum_{n=0}^{\infty}\left(\frac{2(-1)^n}{2n+2}\right)x^{2n+2} = \tan^{-1}(x^2)$. However, we know $f(0) = \tan^{-1}(0) = 0 = C + 0 \therefore C = 0$. Hence,

$$\int \tan^{-1}(x^2)\,dx = \int\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{2n+2}\right)dx$$

$$= \boxed{C + \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(2n+3)}x^{2n+3}}$$

§12.10 #59 Find first 3 nontrivial terms for $e^{-x^2}\cos(x)$ via multiplication of known series

$$e^{-x^2}\cos(x) = \left(1 - x^2 + \tfrac{1}{2}(-x^2)^2 + \cdots\right)\left(1 - \tfrac{1}{2}x^2 + \tfrac{1}{4!}x^4 + \cdots\right)$$

$$= \left(1 - x^2 + \tfrac{1}{2}x^4 + \cdots\right)\left(1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 + \cdots\right)$$

$$= 1 - \tfrac{1}{2}x^2 - x^2 + \tfrac{1}{2}x^4 + \tfrac{1}{2}x^4 + \tfrac{1}{24}x^4 + \cdots$$

$$= \boxed{1 - \tfrac{3}{2}x^2 + \tfrac{25}{24}x^4 + \cdots}$$