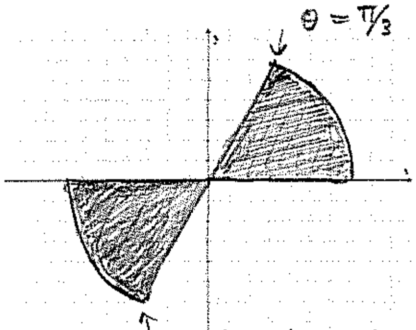
(worth 32pts to help make-up for Test 2 troubles)

Math 132, Quiz 2, Spring 2011 Name:

Credit will be awarded for correct content and clarity of presentation. Prepare for math battle. This Quiz has at least 16 points.

1. Let r, θ be polar coordinates. Complete the following tasks:

a.) sketch the region(s) in the plane which has $-1 \leq r \leq 1$, and $0 \leq \theta \leq \pi/3$



b.) Convert $\underbrace{\tan(\theta)} = 1$ to Cartesian coordinates and identify the curve.

$$\frac{\sin\theta}{\cos\theta} = 1 \longrightarrow \sin\theta = \cos\theta$$
$$\longrightarrow r\sin\theta = r\cos\theta \quad \therefore \boxed{y = x \text{ it's a line}}$$

c.) Find the Cartesian coordinate form of the equation $r = 2\sin(\theta)$,

$$r^2 = 2r\sin\theta$$
$$\underbrace{x^2+y^2 = 2y} \Rightarrow \boxed{x^2+(y-1)^2 = 1}$$

also ok since I didn't say to simplify.

d.) Find the polar radius and standard angle for the point (-2, -3).

$$r = \sqrt{(-2)^2+(-3)^2} = \sqrt{4+9} = \boxed{\sqrt{13} = r}$$

$$\theta = \tan^{-1}\left(\frac{3}{2}\right) + \pi = \underline{4.124 \text{ radians}}$$

or $\boxed{236.3° = \theta}$

($\theta = -123.7°$ also ok)
etc...

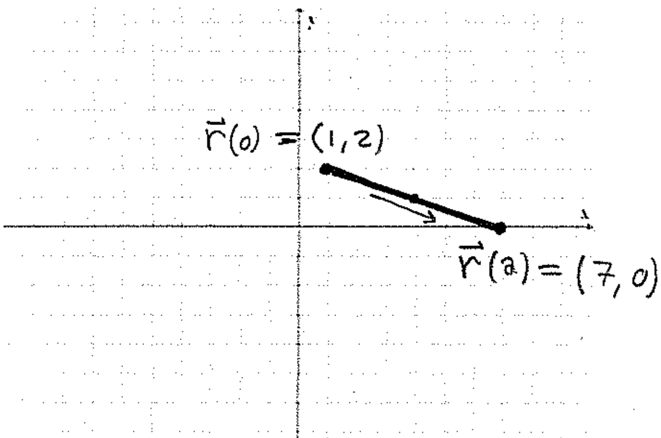
2. Eliminate the parameter t to find the corresponding Cartesian equation for the curves described parametrically below. Also, give inequalities for x and/or y if the Cartesian equation you find describes more than the given curve (tell me which part of the Cartesian curve is covered by the parametrization). Graph the curve and indicate the direction.

a.) $x = 1 + 3t,\ y = 2 - t$ where $0 \leq t \leq 2$.

Comment added by me

$\vec{r}(t) = \langle 1+3t, 2-t \rangle$
$= \underbrace{\langle 1,2 \rangle + t\langle 3,-1 \rangle}$

line with direction $\langle 3,-1 \rangle$.



Cartesian eq^n not hard to find;

$t = \frac{x-1}{3} = 2 - y$

$y = \frac{-1}{3}x + \frac{1}{3} + 2$

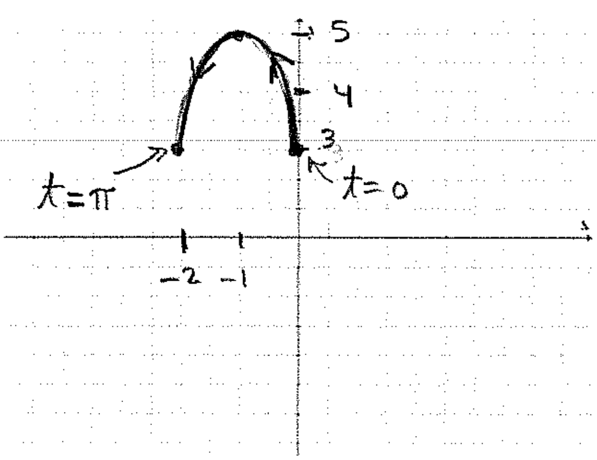
$\boxed{y = \frac{-1}{3}x + \frac{7}{3}}$

(could use different scale, but graph should be a line with direction indicated)

for $1 \leq x \leq 7$
hence $0 \leq y \leq 2$

b.) $x = -1 + \cos(t),\ y = 3 + 2\sin(t)$ where $0 \leq t \leq \pi$.

$\cos t = x + 1$
$\sin t = \frac{y-3}{2}$



$\cos^2 t + \sin^2 t = 1$

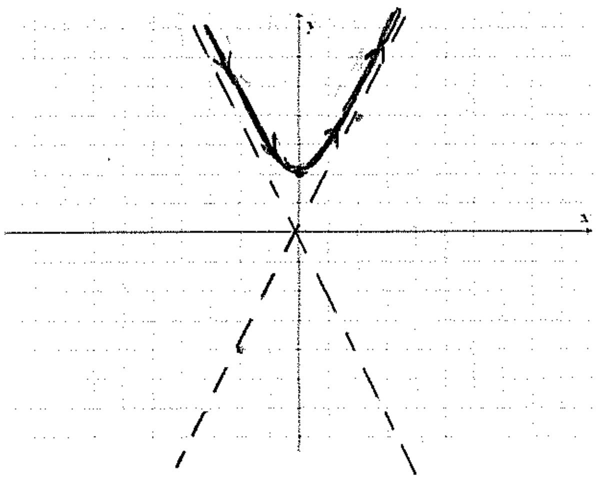
$\boxed{(x+1)^2 + \frac{1}{4}(y-3)^2 = 1}$

Note $0 \leq t \leq \pi$ implies $\cos t \in [-1,1]$ whereas $\sin t \in [0,1]$.

$\therefore$ $\boxed{-2 \leq x \leq 0}$ and $\boxed{3 \leq y \leq 5}$

(look at $x = -1 + \cos t$ and $y = 3 + 2\sin t$)

c.) $x = \sinh(t), \ y = 2\cosh(t)$ where $t \in \mathbb{R}$.



$\cosh^2 t - \sinh^2 t = 1$

$\left(\frac{y}{2}\right)^2 - x^2 = 1$

for $|x|, |y| >> 0$

$y^2/4 \approx x^2 \rightarrow y = \pm 2x$

$\boxed{\frac{y^2}{4} - x^2 = 1}$ for $\boxed{y \geq 2}$

4. Show that the circumference of a circle of radius R is precisely $2\pi R$. *(you should derive this fact by parametrizing the circle then calculating the arclength integral for the full circle. You may assume the circle is at the origin for convenience)*

Circle is parametrized by $x = R\cos t$, $y = R\sin t$ for $0 \leq t \leq 2\pi$. This is easily verified in any number of ways ($x^2+y^2 = R^2\cos^2 t + R^2\sin^2 t = R^2$ and if $0 \leq t \leq 2\pi \Rightarrow \sin t, \cos t$ cover all of $[-1,1]$ hence the circle is covered by these eq^{ns}.)

$$S = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(-R\sin t)^2 + (R\cos t)^2}\, dt$$

$$= \int_0^{2\pi} R\, dt$$

$$= Rt\Big|_0^{2\pi}$$

$$= \boxed{2\pi R.}$$

5. Suppose that $\vec{v}(t) = \langle at,\ be^{ct} \rangle$ denotes the velocity of some particle. Find the (i.) acceleration and (ii.)position as a function of time given that the particle is initially at the origin at time zero. Also, (iii.) set-up an integral which calculates the distance traveled from time zero to time t (DO NOT CALCULATE THE ANTIDERIVATIVE FOR THIS INTEGRAL).

(i.) $\vec{a} = \frac{d\vec{v}}{dt} = \left\langle \frac{d}{dt}(at), \frac{d}{dt}(be^{ct}) \right\rangle = \boxed{\langle a,\ bce^{ct} \rangle = \vec{a}(t)}$

(ii.) $\vec{v}(t) = \frac{d\vec{r}}{dt} = \langle at, be^{ct} \rangle$

$\Rightarrow \vec{r}(t) = \langle \frac{1}{2}at^2 + c_1,\ \frac{b}{c}e^{ct} + c_2 \rangle$

$\vec{r}(0) = \langle 0,0 \rangle = \langle c_1,\ \frac{b}{c} + c_2 \rangle \rightsquigarrow c_1 = 0$, $c_2 = -b/c$

$$\boxed{\vec{r}(t) = \langle \tfrac{1}{2}at^2,\ \tfrac{b}{c}(e^{ct} - 1) \rangle}$$

(iii.) $$\boxed{s(t) = \int_0^t \sqrt{(a\tau)^2 + (be^{c\tau})^2}\, d\tau}$$

6.) Find the parametric equations for a line which goes from (1,2) to (4,19). Write your answer as (i.) a set of scalar equations, (ii.) a vector equation. For both state explicitly the domain for the parameter.

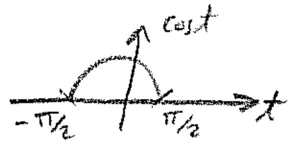
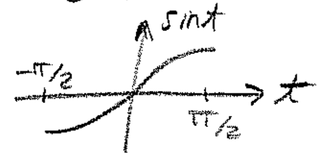
$\overbrace{\vec{r}(t) = (1,2) + t(4-1,\ 19-2)}^{\text{standard technique. Learn this.}} = \langle 1+3t, 2+17t \rangle$

Note $\vec{r}(0) = \langle 1,2 \rangle$ and $\vec{r}(1) = \langle 4, 19 \rangle$ as desired.

(i.) $\boxed{\begin{aligned} x &= 1+3t \\ y &= 2+17t \quad \text{for } 0 \le t \le 1 \end{aligned}}$

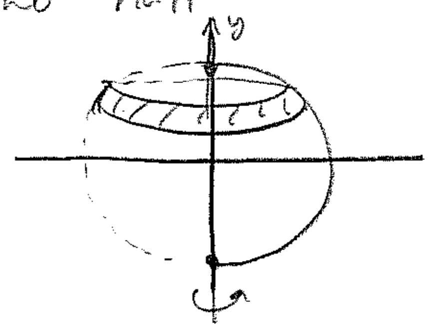
(ii.) $\boxed{\vec{r}(t) = \langle 1+3t,\ 2+17t \rangle \text{ for } 0 \le t \le 1}$

7. Let R be a positive constant. Let $x = R\cos(t),\ y = R\sin(t)$ with $-\pi/2 \le t \le \pi/2$ define a curve C. Find the surface area of the surface of revolution formed by rotating C around the y-axis. (just set-up the integral, don't finish the calculation)

Note $\underline{\cos t}$  and $\underline{\sin t}$ 

$\hookrightarrow 0 \le x \le R$ $\qquad -R \le y \le R$

$x^2 + y^2 = R^2$ so we have $\frac{1}{2}$ a circle, just the right half



$$dA = 2\pi r\, ds$$
$$= 2\pi x \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$
$$= 2\pi R\cos t \sqrt{(-R\sin t)^2 + (R\cos t)^2}\, dt$$
$$= 2\pi R^2 \cos t\, dt$$

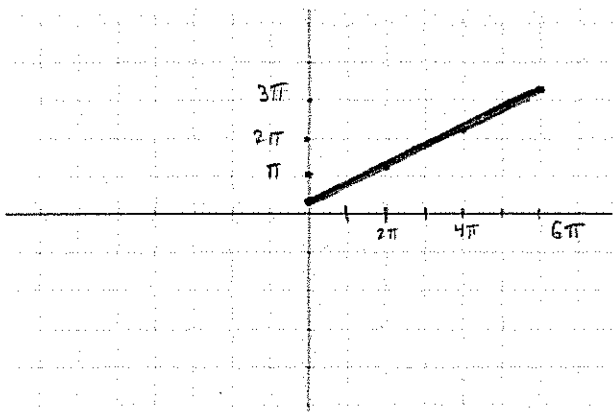
$$\boxed{S.A. = \int_{-\pi/2}^{\pi/2} 2\pi R^2 \cos t\, dt} = \underbrace{4\pi R^2}_{\text{surface area of sphere.}}$$

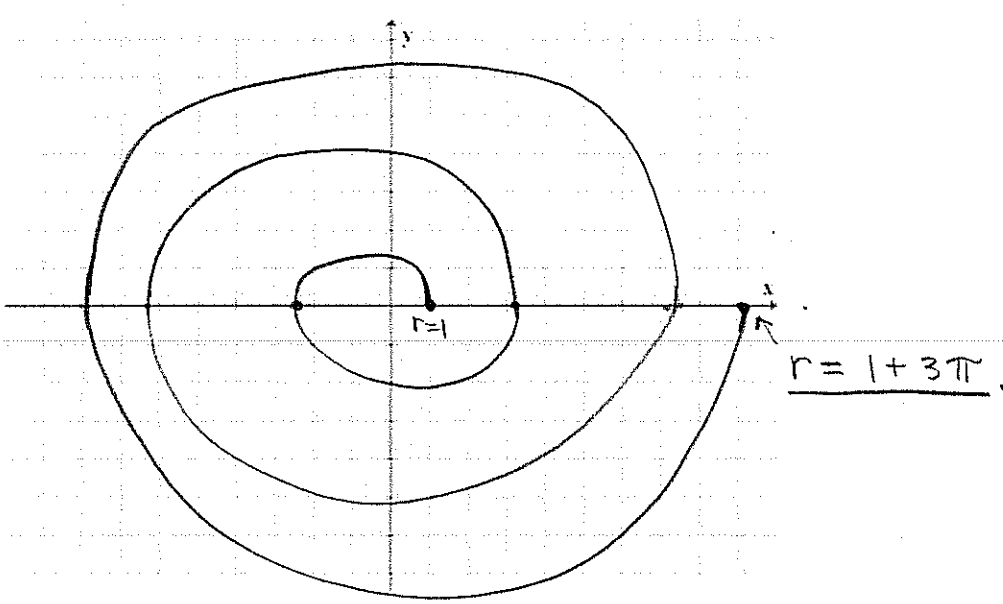
8. Calculate the work done by a force $F(x) = -kx$ during a displacement from $x = 1$ to $x = 3$. Here we consider k a constant.

$$dW = F\,dx = -kx\,dx$$

$$W = \int_1^3 -kx\,dx$$
$$= -\frac{1}{2}kx^2\Big|_1^3$$
$$= -\frac{1}{2}k(9-1) = \boxed{-4k}$$

9. Let r, θ be polar coordinates. Graph the curve defined by $r = 1 + \theta/2$ for $0 \leq \theta \leq 6\pi$. To accomplish this goal you should begin by graphing $r = 1 + \theta/2$ in the "polar plane". You then can transform that graph to create the graph in the xy-plane (which I provide for your convenience, feel free to set the scale, those boxes need not be one unit unless that is appropriate to the graph).





10. Find the equation of the tangent line at (1,1) for the path with parametric equations $x = \cosh(t) + t, \quad y = t^2 + 1$. Where does this path have vertical tangents? (you may leave your answer as the solution to a particular transcendental equation)

$$\vec{r}(t) = \langle \cosh t + t,\ t^2 + 1 \rangle$$

$$\frac{d\vec{r}}{dt} = \langle \sinh t + 1,\ 2t \rangle$$

$$\vec{r}(t) = \langle \cosh t + t,\ t^2 + 1 \rangle = \langle 1, 1 \rangle \Rightarrow \begin{cases} \cosh t + t = 1 \\ t^2 + 1 = 1 \end{cases}$$

$$t = 0$$

We're finding tangent at $t = 0$

$$\boxed{\vec{l}(t) = (1,1) + t\langle 1, 0 \rangle}$$

(vector-parametric eq$^{\underline{n}}$ of tangent line. alternatively $x = 1 + t$ and $y = 1$)

<u>vertical tangents</u>: $\frac{dx}{dt} = 0$

$$\sinh t + 1 = 0$$

vertical tangent at $\underline{t = \sinh^{-1}(-1)}$ $\left(\cosh(\sinh^{-1}(-1)) + \sinh^{-1}(-1),\ [\sinh^{-1}(-1)]^2 + 1\right)$

11. Suppose that C is the curve which is the solution set of the equation $x^2 + y^2 - 2xy = 34$. Find a function F of (x,y) such that the inverse image of 4 gives the curve C. In other words, tell me a formula for $F(x,y)$ such that $F^{-1}(\{4\}) = C$.

$$\underbrace{x^2 + y^2 - 2xy - 30}_{F(x,y)} = 4$$

$$\boxed{F(x,y) = x^2 + y^2 - 2xy - 30}$$

12. For the curve C in the previous problem, is it possible to find a function $f : U \subseteq \mathbb{R} \to \mathbb{R}$ such that $graph(f) = C$? (Explain with mathematical analysis)

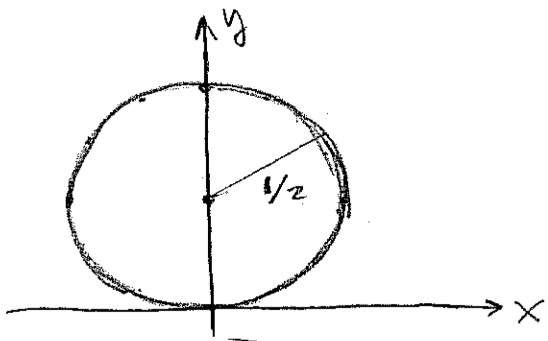
No, the given curve fails the vertical line test.

Note $x^2 + y^2 - 2xy = (x - y)^2 = 34 \Rightarrow x - y = \pm\sqrt{34}$

Hence $y = x \pm \sqrt{34}$ (need <u>two</u> functions to cover the <u>whole</u> curve C.)

13. Find the area bounded by $r = \sin(\theta)$.

$$r^2 = r\sin\theta \longrightarrow x^2 + y^2 = y$$



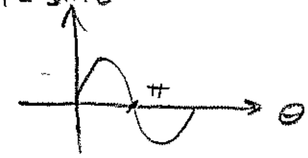
$$\boxed{A = \pi/4}$$

$$x^2 + (y - 1/2)^2 = 1/4$$

circle at $(0, 1/2)$ with radius $1/2$.

(also easy to see from

$r = \sin\theta$



Of course we can calculate this,

$$A = \int_0^{\pi} \frac{1}{2} \sin^2\theta \, d\theta$$

$$= \int_0^{\pi} \frac{1}{4} \left(1 - \cos(2\theta)\right) d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_0^{\pi} = \boxed{\frac{\pi}{4}}$$

14. Derive the formula for $\frac{d^2y}{dx^2}$ given that $x = x(t)$ and $y = y(t)$. You should use the result we found before; namely that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. It may be helpful to use the notation $x = f(t)$ and $y = g(t)$ etc...

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{d}{dt}(x)} = \frac{1}{\frac{dx}{dt}} \frac{d}{dt}\left[\frac{dy/dt}{dx/dt}\right]$$

$$= \frac{1}{\frac{dx}{dt}}\left[\frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2}\right]$$

$$= \boxed{\frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^3}}$$

$\dot{x} = \frac{dx}{dt}$

$\dot{y} = \frac{dy}{dt}$

$\ddot{x} = \frac{d^2x}{dt^2}$

etc..

a.k.a. $$\frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$